1. (a) Let $X_n$ and $Y_n$ be independent normal distributions. Then $X_n \to_d X_1$ and $Y_n \to_d Y_1$ but $X_n + Y_n \sim N(0,2)$ does not converge to $X_1 - X_1 = 0$.

(b) The characteristic function of $(X_n, Y_n)$ is
\[
\phi(t_1, t_2) = E[\exp\{it_1 X_n + it_2 Y_n\}] = E[\exp\{it_1 X_n\}]E[\exp\{it_2 Y_n\}]
\]
\[\to_d E[\exp\{it_1 X\}]E[\exp\{it_2 Y\}] = E[\exp\{it_1 X + it_2 Y\}].\]

Thus, $(X_n, Y_n) \to_d (X, Y)$. Since $(x, y) \to x + y$ is continuous, $X_n + Y_n \to_d X + Y$ follows from the continuous mapping theorem.

2. (a) This follows from $E[\sum_{i=1}^n w_i Y_i / x_i] = \sum_{i=1}^n w_i E[Y_i] / x_i = \beta$.

(b) The variance is given as
\[
\sum_{i=1}^n w_i^2 x_i / x_i^2 = \sum_{i=1}^n w_i^2 / x_i.
\]

We minimize the above term subject to constraint $\sum_{i=1}^n w_i = 1$. The Lagrange-multiplier gives
\[w_i = x_i / \sum_{j=1}^n x_j.\]

(c) For the optimal weights, we have
\[
\sum_{i=1}^n w_i Y_i / x_i = \beta + \frac{\sum_{i=1}^n \epsilon_i \sqrt{x_i}}{\sum_{i=1}^n x_i}.
\]

Since $\max x_i / \sum_{j=1}^n x_j \to 0$, by the weighted CLT,
\[
\sqrt{\sum_{i=1}^n x_i} \to_d N(0,1).
\]

Thus,
\[
\{\sum_{i=1}^n x_i\}^{1/2} \left[ \sum_{i=1}^n w_i Y_i / x_i - \beta \right] \to_d N(0,1).
\]