1. Note that \( X_i = \beta X_{i-1} + \varepsilon_i = \ldots = \beta^{i-1} \varepsilon_1 + \beta^{i-2} \varepsilon_2 + \ldots + \beta^0 \varepsilon_i \). Hence

\[
X_n = \frac{1}{n} \sum_{i=1}^{n} (\beta^{i-1} \varepsilon_1 + \beta^{i-2} \varepsilon_2 + \ldots + \beta^0 \varepsilon_i) \\
= \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sum_{j=i}^{n} \beta^{j-i} \\
= \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sum_{j=0}^{n-i} \beta^j \\
= \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \left( \frac{1 - \beta^{n-i+1}}{1 - \beta} \right)
\]

2. Note that for odd \( n \), \((-1)^{n-i+1} = -1\) for odd \( i \)-s and \((-1)^{n-i+1} = 1\) for even \( i \)-s. Subsisting in Equation 1 for odd \( n \) we have

\[
X_n = \frac{1}{2n} \sum_{i=1}^{n} \varepsilon_i - \frac{1}{2n} \left( \sum_{i=1,3,...,n} (-1) \cdot \varepsilon_i + \sum_{i=2,4,...,n-1} \varepsilon_i \right) \\
= \frac{2}{2n} \sum_{i=1,3,...,n} \varepsilon_i
\]

The proof for even \( n \) is similar.

3. First note that \( E|\varepsilon_i| < \infty \). This follows since \( E|\varepsilon_i| \leq E[\varepsilon^2 + 1] = (\sigma^2 + \mu^2) + 1 \). Hence for every fixed \( \delta > 0 \)

\[
P \left( n^\alpha \left| \frac{1}{n(1-\beta)} \sum_{i=1}^{n} \varepsilon_i \beta^{n-i+1} \right| > \delta \right) \leq \frac{n^\alpha}{\delta} E \left[ \left| \frac{1}{n(1-\beta)} \sum_{i=1}^{n} \varepsilon_i \beta^{n-i+1} \right| \right] \quad \text{(Markov inequality)} \\
\leq \frac{n^\alpha}{n(1-\beta)\varepsilon} \sum_{i=1}^{n} \beta^{n-i} E[|\varepsilon_i|] \quad \text{(triangle inequality)} \\
\leq n^{\alpha-1} E|\varepsilon_i| \delta \to 0 \quad \text{for } \alpha < 1
\]

4. By Equation 1, we have that

\[
X_n = \frac{1}{n(1-\beta)} \sum_{i=1}^{n} \varepsilon_i - \frac{1}{n(1-\beta)} \sum_{i=1}^{n} \varepsilon_i \beta^{n-i+1}.
\]

Since \( \varepsilon_i \) are i.i.d. with mean \( \mu \), we obtain from the w.l.l.n that

\[
\frac{1}{n(1-\beta)} \sum_{i=1}^{n} \varepsilon_i \to_p \frac{\mu}{1-\beta}.
\]

From the previous question we obtain that

\[
\frac{1}{n(1-\beta)} \sum_{i=1}^{n} \varepsilon_i \beta^{n-i+1} \to_p 0.
\]
By Slutsky’s Theorem we conclude that $X_n \to_d \mu/(1 - \beta) + 0$ which is equivalent to $X_n \to_p \mu/(1 - \beta)$.

5. By Equation 1 we can write

$$\sqrt{n} \left(X_n - \frac{\mu}{1 - \beta}\right) = \sqrt{n} \left(\frac{1}{n(1 - \beta)} \sum_{i=1}^{n} \varepsilon_i - \frac{\mu}{1 - \beta}\right) - \sqrt{n} \left(\frac{1}{n(1 - \beta)} \sum_{i=1}^{n} \varepsilon_i \beta^{n-i+1}\right)$$

Note that $\varepsilon_i/(1 - \beta)$ are i.i.d. and have mean $\mu/(1 - \beta)$ and variance $\sigma^2/(1 - \beta)^2$. Hence by the CLT we obtain that

$$\sqrt{n} \left(\frac{1}{n(1 - \beta)} \sum_{i=1}^{n} \varepsilon_i - \frac{\mu}{1 - \beta}\right) \to_d N(0, \sigma^2/(1 - \beta)^2).$$

By Question 3, we have that

$$\sqrt{n} \left(\frac{1}{n(1 - \beta)} \sum_{i=1}^{n} \varepsilon_i \beta^{n-i+1}\right) \to_p 0$$

and the result follows from Slutsky’s Theorem.

6. By Question 2, for even $n = 2k$ we have

$$X_n = \frac{1}{2k} \sum_{i=1}^{k} \varepsilon_{2i} = \frac{1}{k} \sum_{i=1}^{k} \varepsilon_{2i} \to_{a.e.} \frac{\mu}{2}.$$

Note that this is sum of i.i.d. random variables with expectation $\mu/2$ and variance $\sigma^2/4$. By the CLT we obtain that

$$\sqrt{n} \left(X_n - \frac{\mu}{2}\right) = \sqrt{k} \sqrt{2} \left(X_n - \frac{\mu}{2}\right) \to_d N \left(0, \frac{\sigma^2}{2}\right)$$

For odd $n = 2k + 1$, note that

$$\frac{1}{n} = \frac{1}{2k + 1} = \frac{1}{2(k + 1)} + \frac{1}{2(k + 1)(2k + 1)}.$$

Hence,

$$X_n = \frac{1}{2k + 1} \sum_{i=1}^{k+1} \varepsilon_{2i-1} = \frac{1}{2(k + 1)} \sum_{i=1}^{k+1} \varepsilon_{2i-1} + \frac{1}{2(k + 1)(2k + 1)} \sum_{i=1}^{k+1} \varepsilon_{2i-1}$$

and since the first expression is sum of i.i.d. random variables and $\frac{1}{2(k+1)(2k+1)} \sum_{i=0}^{k} \varepsilon_{2i-1} \to_p 0$ we obtain that $\sqrt{n}(X_n - \mu/2) \to_d N(0, \sigma^2/2)$.

7. Write $g(t) = t^2$ and note that $g'(t) = 2t$. By the delta method we have that

$$\sqrt{n} \left(g(X_n) - g \left(\frac{\mu}{1 - \beta}\right)\right) \to_d g' \left(\frac{\mu}{1 - \beta}\right) N \left(0, \frac{\sigma^2}{(1 - \beta)^2}\right)$$

$$= N \left(0, \frac{4\mu^2\sigma^2}{(1 - \beta)^4}\right).$$
8. By Question 7 we see that $\sqrt{n}(X_n)^2$ converges to 0. By Question 5,

$$\frac{(1 - \beta)}{\sigma} \sqrt{nX_n} \rightarrow_d N(0, 1).$$

Using the continuous mapping theorem we conclude that $n(1 - \beta)^2 \sigma^{-2}(X_n)^2$ converges to the square of standard normal, i.e., to $\chi_1^2$. 