BIOS 760: Midterm II 2010

Suppose that \( \varepsilon_1, \varepsilon_2, \ldots \) are i.i.d. random variables with mean \( \mu \) and variance \( \sigma^2 \). Define \( X_n \) as the autoregressive model sequence \( X_1 = \varepsilon_1 \), and for \( n \geq 2 \),

\[
X_n = \beta X_{n-1} + \varepsilon_n,
\]

where \(-1 \leq \beta < 1\).

1. (4 points) Denote \( \overline{X}_n \equiv \frac{1}{n} \sum_{i=1}^{n} X_i \). Use change of order in the summation to show that

\[
\overline{X}_n = \frac{1}{n(1-\beta)} \sum_{i=1}^{n} \varepsilon_i - \frac{1}{n(1-\beta)} \sum_{i=1}^{n} \varepsilon_i \beta^{n-i+1}.
\]

(1)

Hint: First, show that \( X_i = \varepsilon_i + \beta \varepsilon_{i-1} + \ldots + \beta^{i-1} \varepsilon_1 \).

2. (3 points) Conclude that for \( \beta = -1 \): \( \overline{X}_n = n^{-1}(\varepsilon_1 + \varepsilon_3 + \ldots + \varepsilon_n) \) for odd \( n \), and \( \overline{X}_n = n^{-1}(\varepsilon_2 + \varepsilon_4 + \ldots + \varepsilon_n) \) for even \( n \).

3. (5 bonus points) For \(-1 < \beta < 1\), show that

\[
\frac{1}{n(1-\beta)} \sum_{i=1}^{n} \varepsilon_i \beta^{n-i} = o_p(n^{-\alpha}),
\]

for every \( 0 \leq \alpha < 1 \). Hint: You can use the Markov inequality.

4. (5 points) For \(-1 < \beta < 1\), show that \( \overline{X}_n \rightarrow_p \frac{\mu}{1-\beta} \). Hint: Use Equation (1), the previous question, and the fact that \( o_p(n^{-\alpha}) = o_p(1) \) for every \( \alpha \geq 0 \).

5. (4 points) For \(-1 < \beta < 1\), show that

\[
\sqrt{n} \left( \overline{X}_n - \frac{\mu}{1-\beta} \right) \rightarrow_d N(0, \tau^2),
\]

and derive \( \tau^2 \).

6. (5 bonus points) For \( \beta = -1 \), show that

\[
\sqrt{n} \left( \overline{X}_n - \frac{\mu}{2} \right) \rightarrow_d N(0, \nu^2),
\]

and derive \( \nu^2 \). Hint: Use Question 2. Note the difference between odd and even \( n \). Try first to show for even \( n \).

7. (4 points) For \(-1 < \beta < 1\), derive the asymptotic distribution of

\[
\sqrt{n} \left( \overline{X}_n \right)^2 - \frac{\mu^2}{(1-\beta)^2}.
\]

8. (5 points) For \(-1 < \beta < 1\), when \( \mu = 0 \), show that \( \sqrt{n}(\overline{X}_n)^2 \rightarrow_d 0 \) but

\[
\frac{n(1-\beta)^2(\overline{X}_n)^2}{\sigma^2} \rightarrow_d \chi^2_1.
\]

Hint: Use the continuous mapping theorem.