1. Let $X$ be a positive random variable with density:

$$p_\theta(x) = \frac{x}{\theta} e^{-x^2/(2\theta)}, \quad x > 0,$$

where $0 < \theta < \infty$. Do the following:

(a) (3 points) Show that $\{p_\theta\}$ is a one-parameter exponential family with canonical parameter $\eta(\theta) = -1/(2\theta)$.

(b) (3 points) Show that the moment generating function for $T(X) = X^2$ is $M_{T(X)}(t) = (1 - 2\theta t)^{-1}$.

2. Let $X : \Omega \mapsto R$ be a measurable function on the measure space $(\Omega, \mathcal{A}, \mu)$, where $\mu(\Omega) < \infty$ and $|X| < \infty$ almost everywhere, and define

$$Y_n = \left( \frac{X}{1 + |X|} \right)^n.$$ 

Do the following:

(a) (3 points) Show that $Y_n \to_{a.e.} 0$ and $Y_n \to_{\mu} 0$.

(b) (4 points) Show that $\int Y_n d\mu \to 0$.

3. Let $(X_1, Y_1)$ and $(X_2, Y_2)$ be two independent pairs of random variables with $(X_j, Y_j)$ being bivariate normal with mean zero and covariance

$$\Sigma = \begin{pmatrix} 4 & 2\rho \\ 2\rho & 1 \end{pmatrix},$$

for $j = 1, 2$, where $|\rho| \leq 1$. Let $\mathcal{N}_1$ be the $\sigma$-field generated by $Y_1$ and $Y_2$ (i.e., $\mathcal{N}_1 = \sigma(Y_1, Y_2)$), and let $\mathcal{N}_2$ be the $\sigma$-field generated by $Z = \max(Y_1, Y_2)$ (i.e., $\mathcal{N}_2 = \sigma(Z)$). Do the following:

(a) (3 points) Show that $E[X_1|\mathcal{N}_1] = E[X_1|Y_1] = 2\rho Y_1$.

(b) (4 points) Show that $\mathcal{N}_2 \subset \mathcal{N}_1$.

(c) (5 bonus points) Show that

$$E [X_1|\mathcal{N}_2] = \rho \left( Z - \frac{\phi(Z)}{\Phi(Z)} \right),$$

where $\phi(x)$ is the standard normal density at $x$ and $\Phi(x) = \int_{-\infty}^x \phi(u)du$. 
