1. Let $X = (X_1, X_2)$ be a bivariate random variable with density function

$$f(x_1, x_2) = 2, \quad 0 < x_1 < x_2 < 1.$$ 

(a) Note that $F_X(a, b) = 0$ if either $a < 0$ or $b < 0$.

Assume that $0 < a, b < 1$, then

$$F_X(a, b) = \int_0^b \int_0^a f(x_1, x_2) dx_1 dx_2 = \int_0^b \int_0^\min(a, x_2) 2 dx_1 dx_2$$

$$= \int_0^b 2 \min(a, x_2) dx_2 = \int_0^b (2x_2 - 2(x_2 - a)_+) dx_2$$

$$= b^2 - (b - a)^2_+$$

For $a \geq 1$ we have $F_X(a, b) = F_X(1, b)$ and similarly for $b \geq 1$ we have $F_X(a, b) = F_X(a, 1)$

(b) For $0 < x_1 < 1$,

$$f(x_1) = \int f(x_1, x_2) dx_2 = \int_{x_1}^1 2 dx_2 = 2(1 - x_1).$$

For $0 < x_2 < 1$,

$$f(x_2) = \int f(x_1, x_2) dx_1 = \int_0^{x_2} 2 dx_1 = 2x_2.$$

(c) No, since $f(x_1, x_2) \neq f(x_1) f(x_2)$ we conclude that $X_1$ and $X_2$ are dependent.

(d) For $0 < x_1 < x_2$,

$$f(x_1 | x_2) = \frac{f(x_1, x_2)}{f(x_2)} = \frac{2}{2x_2} = \frac{1}{x_2}.$$ 

Hence

$$E[X_1 | X_2] = \int_0^{x_2} \frac{x_1}{x_2} dx_1 = \frac{x_2}{2}.$$

2. Let $X = (X_1, X_2)$ be as in Question 1.

Define the function $u(x_1, x_2) = \left( \frac{x_1}{x_2}, x_2 \right)$ and denote $Y = (Y_1, Y_2) = u(X_1, X_2)$.

(a) The Jacobian matrix is the matrix of first-order partial derivatives.

$$J_u = \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{pmatrix} 1/x_2 & -1/(x_2)^2 \\ 0 & 1 \end{pmatrix}.$$ 

The Jacobian is the determinant of the Jacobian matrix and it equals $1/x_2$.  

(b) The density of $Y$ can be computed from the density of $X$ as follows:

$$ f_Y(y_1, y_2) = f_X(u^{-1}(y_1, y_2)) J_u^{-1}. $$

Note that $x_2 = y_2$ and $x_1 = y_1 y_2$ and $J_u^{-1} = x_2 = y_2$. Hence,

$$ f_Y(y_1, y_2) = 2y_2 1_{0<y_1 y_2 < y_2 < 1}. $$

Since $0 < y_1 y_2 < y_2 \implies 0 < y_1 < 1$, we can write

$$ f_Y(y_1, y_2) = 2y_2 1_{0<y_1 y_2 < 1}. $$

(c) Yes, $Y_1$ is independent of $Y_2$. To see that note that for $0 < y_1, y_2 < 1$,

$$ f(y_1) = \int_0^1 f(y_1, y_2) dy_2 = \int_0^1 2y_2 dy_2 = 1 $$

$$ f(y_2) = \int_0^1 f(y_1, y_2) dy_1 = \int_0^1 2y_2 dy_1 = 2y_2 $$

Since $f_Y(y_1, y_2) = f(y_1) f(y_2)$ we conclude that $Y_1$ is independent of $Y_2$.

3. Let $X_1, X_2, X_3$ be random variables on some probability space.

(a) Note that since $E[E(X|Y)] = E[X]$, we have

$$ E[X_1 E(X_2|X_3)] = E[E[X_1 E(X_2|X_3)]|X_3]. $$

Since $E(X_2|X_3)$ is measurable with respect to the $\sigma$-field generated by $X_3$,

$$ E[E[X_1 E(X_2|X_3)]|X_3] = E[E(X_2|X_3) E(X_1|X_3)]. $$

Similarly, $E[X_2 E(X_1|X_3)] = E[E(X_2|X_3) E(X_1|X_3)]$, and we obtain the required equality.

(b) Let

$$ \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{pmatrix}. $$

Note that

$$ E[E(X_1|X_2)|X_3] = E[\Sigma_{12} \Sigma_{22}^{-1} X_2|X_3] = \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{23} \Sigma_{33}^{-1} X_3 $$

$$ E[E(X_2|X_1)|X_3] = E[\Sigma_{21} \Sigma_{11}^{-1} X_1|X_3] = \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{13} \Sigma_{33}^{-1} X_3. $$

Thus when $\Sigma_{22}^{-1} \Sigma_{23} \neq \Sigma_{11}^{-1} \Sigma_{13}$, we also have $E(X_1|X_2)|X_3 \neq E[E(X_2|X_1)|X_3].$
(c) Assume that $X_1, X_2, X_3$ are i.i.d. and that $E[|X_1|] < \infty$.

$$E[X_1|X_1 + X_2 + X_3] = \frac{X_1 + X_2 + X_3}{3}$$

Since $X_1, X_2, X_3$ are i.i.d., $E[X_1|X_1 + X_2 + X_3] = E[X_2|X_1 + X_2 + X_3] = E[X_3|X_1 + X_2 + X_3]$. Thus

$$E[X_1|X_1 + X_2 + X_3] = \frac{\sum_{i=1}^{3} E[X_i|X_1 + X_2 + X_3]}{3}$$
$$= \frac{E[X_1 + X_2 + X_3|X_1 + X_2 + X_3]}{3}$$
$$= \frac{X_1 + X_2 + X_3}{3},$$

where the last inequality follows since $X_1 + X_2 + X_3$ is measurable with respect to the $\sigma$-field generated by itself.