* $X, Y \sim$ iid Exp(1)

$f_X(t) = f_Y(t) = e^{-t} \qquad t>0.$

$F_X(t) = F_Y(t) = P(X \leq t) = 1 - e^{-t} \qquad t>0.$

$P(X>t) = e^{-t} \qquad t>0.$

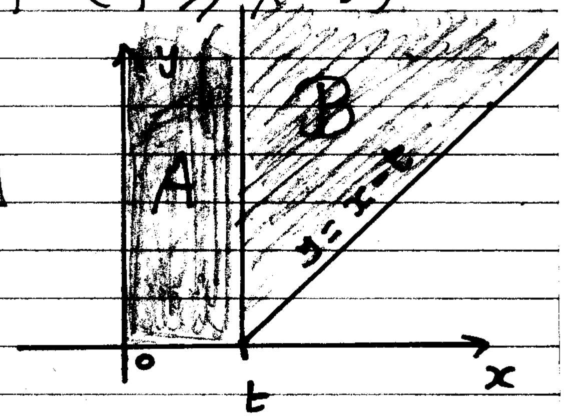
* Find the pdf of $T := X - Y$.

- The support of T is $(-\infty, \infty)$.
- Take $t>0$, then $F_T(t)$

$= P(T \leq t) = P(X - Y \leq t) = P(Y \geq X - t)$

$$\boxed{f_{X,Y}(x,y) = e^{-x} e^{-y} \qquad x, y > 0}$$

$= P(A) + P(B)$



- $P(A) = P(X \leq t) = 1 - e^{-t}$.
- For $P(B)$ we integrate over x from t to ∞, and (at each x) over y from $x-t$ to ∞,

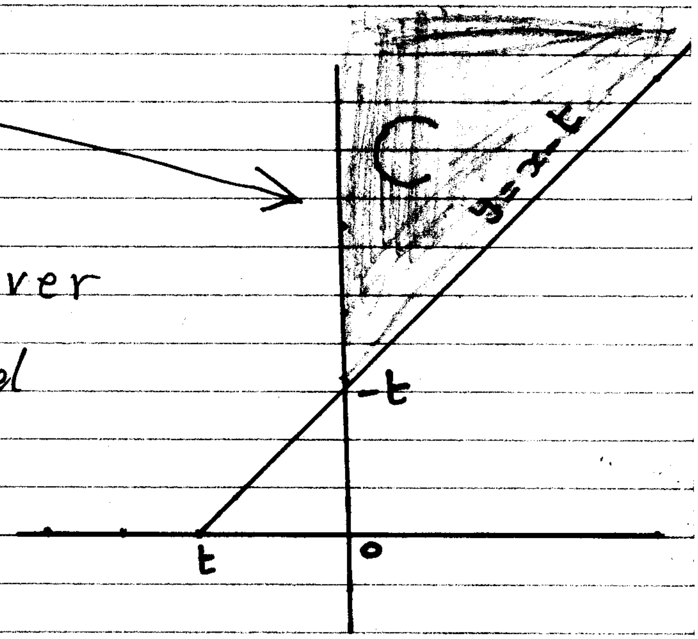
$$P(B) = \int_{x=t}^{\infty} \int_{y=x-t}^{\infty} e^{-x} e^{-y}\, dy\, dx$$

$$= \int_t^{\infty} e^{-x} \left[\int_{x-t}^{\infty} e^{-y}\, dy \right] dx = \int_t^{\infty} e^{-x} e^{-(x-t)}\, dx$$

$$= e^{t} \int_t^{\infty} e^{-2x}\, dx = \frac{1}{2} e^{-t}.$$

- $F_T(t) = P(A) + P(B) = 1 - \frac{1}{2}e^{-t}$ $\quad t > 0.$

- For $t < 0$,
$P(T \le t) = P(C)$,
so we integrate over x from 0 to ∞ and (at each x) over y from $x - t$ to ∞.



$$P(C) = \int_{x=0}^{\infty} \int_{y=x-t}^{\infty} e^{-x} e^{-y} \, dy \, dx = \frac{1}{2} e^{+t}.$$

- $$F_T(t) = \begin{cases} 1 - \frac{1}{2} e^{-t} & t > 0, \\ \frac{1}{2} e^{t} & t \le 0. \end{cases}$$

- $$f_T(t) = \begin{cases} \frac{1}{2} e^{-t} & t > 0, \\ \frac{1}{2} e^{t} & t \le 0. \end{cases}$$
$$= \frac{1}{2} e^{-|t|} \qquad -\infty < t < \infty.$$

☆ Another way to compute F_T:

- For $t>0$, $P(T \le t) = P(X-Y \le t)$

$= P_{X,Y}(Y \geq X-t)$

$= E_X\left[P_{Y|X}(Y \geq X-t \mid X) \right]$ $\boxed{\text{by double expectation}}$

- $P_{Y|X}(Y \geq X-t \mid X) = \begin{cases} 1 & \text{if } X \le t \\ e^{-(X-t)} & \text{if } X > t \end{cases}$

$= I(X \le t) + I(X > t) \cdot \exp\{-(X-t)\}$.

Now take expectations,

- $E_X[I(X \le t)] = P(X \le t) = -e^{-t} + 1$.
- $E_X\left[I(X>t) \cdot \exp\{-(X-t)\} \right]$

$= \int_t^\infty e^{-(x-t)} e^{-x} \, dx = \frac{1}{2} e^{-t}$.

- This gives $P(T \le t) = 1 - e^{-t} + \frac{1}{2} e^{-t} = 1 - \frac{1}{2} e^{-t}$, $t > 0$.

- For $t < 0$, a similar derivation gives $F_T(t) = \frac{1}{2} e^{t}$ $\quad t < 0$.