BIOS 660, the basics

* Sets and events

* Set: $\{y : y < 5\}$ or $(-\infty, 5)$

* Event: $\{X < 5\}$ or $\{X \text{ is less than } 5\}$ or $\{X \in (-\infty, 5)\}$

* $P(\text{event})$
$P(X < 5)$ is shorthand for $P(\{X < 5\})$

* The sample space

* If B is true whenever A is true, we say that A implies B; in symbols: $A \implies B$. In the Venn diagram, A would be completely inside B

* If A implies B then A is a subset of or equal to B, written $A \subseteq B$

* If A implies B and B implies A, then $A = B$

* If $A = B$ then $P(A) = P(B)$. The opposite and the converse are not true. $A \neq B$ does not imply that $P(A) \neq P(B)$. $P(A) = P(B)$ does not imply that $A = B$

* $\#A$ or $\#(A)$ is the number of elements in A, the cardinality of A

* Inclusion-exclusion (IE) principle for two events:
$\#(A \cup B) = \#A + \#B - \#(A \cap B)$

* IE for 3 events:
$\#(A \cup B \cup C) = \#A + \#B + \#C - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + \#(A \cap B \cap C)$
Notice how the signs (of the intersections) alternate

* IE in general, in English:
$\#(A_1 \cup A_2 \cup ... \cup A_n)$ = sum of all events − sum of all intersections of 2 events + sum of all intersections of 3 events − ... + ... − ... + ...the intersection of the n events

* IE for probabilities: replace $\#()$ by $P()$

* $I(A)$ is the indicator variable of event A
* $I(A) = 1$ if A is true
* $I(A) = 0$ if A is false

* $P(A \cap B) = P(B)P(A|B)$ implies $P(A|B) = P(A \cap B)/P(B) = P(B|A)P(A)/P(B)$ which implies $P(A|B)/P(B|A) = P(A)/P(B)$
(assuming none of the denominators is zero)

* $P(A \cap B) = P(B)P(A|B)$ implies $P(B) = P(A \cap B)/P(A|B)$. Similar formulae apply to pdf's and pmf's of random variables (X, Y)

* Random variables

* Simplest distributions:
Discrete: Bernoulli (e.g. indicator variables)
Continuous: Uniform

* The support of X, support(X), is the set of x at which the pdf or pmf of X is positive.

* (Univariate) Continuous, discrete, mixed

* Independence

* Functions of independent random variables are independent

* Expectations

* $E[I(A)] = P(A)$

* Expectations everywhere! (variance, covariance, MGF)

* MGF: $M_X(t) := \mathrm{E}[e^{tX}]$

* MGF: Can be used to compute moments

* MGF: Can be used to compute $E[\exp(tY)]$ for a specific t

* The MGF of a linear combination of independent random variables

* Expectations and variances and covariances of linear functions of random variables

* Double expectation for means, probabilities, pdfs and pmfs:

- The marginal expectation is the expected value of the conditional expectation
- The marginal probability is the expected value of the conditional probability
- The marginal pdf/pmf is the expected value of the conditional pdf/pmf
- The marginal variance is the sum of the expected value of the conditional variance plus the variance of the conditional expectation (this breaks the previous pattern)

- The marginal covariance is the sum of the expected value of the conditional covariance plus the covariance of the conditional expectations

* $E[g(X)] \neq g(E[X])$ for all random X unless $g()$ is linear (or constant) on support(X).

* Inequalities: Jensen, Markov, Chebychev, CS

* Transformation of random variables:
Computing the pdf/pmf of $g(X)$:
For discrete.
For continuous: 1) using the cdf, 2) using Jacobians

* Bivariate: Continuous, discrete, mixed

* Bivariate distributions: Joint, marginal, conditional pdf, pmf

* Computing the conditional from the joint and vice versa, for (X, Y) both continuous, both discrete, one continuous and one discrete

* Bivariate (X, Y): both continuous, both discrete, one continuous and one discrete.
Examples: Bivariate normal, bivariate Bernoulli, Poisson-gamma mixture (negative binomial), binomial-beta mixture (beta-binomial), hypergeometric, ...

* Covariance and correlation

* In general, zero covariance (or zero correlation) does not imply independence

* Covariances of linear functions of random variables

* How to check independence: joint = marginal * marginal,
conditional = marginal
Must hold at *every point* in the sample space

The expected value of the product of *independent* random variables is the product of the expected values. That is, if X and Y are independent then $\mathrm{E}[XY] = \mathrm{E}[X]E[Y]$.

If $\mathrm{E}[XY] = \mathrm{E}[X]E[Y]$, then all we can say is that $\mathrm{Cov}(X, Y) = 0$, but there is no guarantee that X and Y are independent. Example: $P(X = -1, Y = -1) = P(X = 1, Y = 1) = P(X = -1, Y = 1) = P(X = 1, Y = -1) = P(X = 0, Y = 0) = 1/5$
Verify that $\mathrm{E}[XY] = \mathrm{E}[X]E[Y]$. Verify that X and Y are not independent.