$x(2x-1)$, we can split the interval $[0, 1]$ into two subintervals, in each of which the integrand is a polynomial. As x varies from 0 to 1, the product $x(2x-1)$ changes sign at the point $x = \frac{1}{2}$; it is negative if $0 < x < \frac{1}{2}$ and positive if $\frac{1}{2} < x < 1$. Therefore, we use the additive property to write

$$\begin{aligned}\int_0^1 |x(2x-1)|\,dx &= -\int_0^{1/2} x(2x-1)\,dx + \int_{1/2}^1 x(2x-1)\,dx\\ &= \int_0^{1/2}(x-2x^2)\,dx + \int_{1/2}^1 (2x^2-x)\,dx\\ &= (\tfrac{1}{8}-\tfrac{1}{12}) + (\tfrac{7}{12}-\tfrac{3}{8}) = \tfrac{1}{4}.\end{aligned}$$

1.26 Exercises

Compute each of the following integrals.

1. $\int_0^3 x^2\,dx.$
2. $\int_{-3}^3 x^2\,dx.$
3. $\int_0^2 4x^3\,dx.$
4. $\int_{-2}^2 4x^3\,dx.$
5. $\int_0^1 5t^4\,dt.$
6. $\int_{-1}^1 5t^4\,dt.$
7. $\int_0^1 (5x^4-4x^3)\,dx.$
8. $\int_{-1}^1 (5x^4-4x^3)\,dx.$
9. $\int_{-1}^2 (t^2+1)\,dt.$
10. $\int_2^3 (3x^2-4x+2)\,dx.$
11. $\int_0^{1/2} (8t^3+6t^2-2t+5)\,dt.$
12. $\int_{-2}^4 (u-1)(u-2)\,du.$
13. $\int_{-1}^0 (x+1)^2\,dx.$
14. $\int_0^{-1} (x+1)^2\,dx.$
15. $\int_0^2 (x-1)(3x-1)\,dx.$
16. $\int_0^2 |(x-1)(3x-1)|\,dx.$
17. $\int_0^3 (2x-5)^3\,dx.$
18. $\int_{-3}^3 (x^2-3)^3\,dx.$
19. $\int_0^5 x^2(x-5)^4\,dx.$
20. $\int_{-2}^{-4} (x+4)^{10}\,dx.$ [*Hint:* Theorem 1.18.]
21. Find all values of c for which

    (a) $\int_0^c x(1-x)\,dx = 0,$ (b) $\int_0^c |x(1-x)|\,dx = 0.$
22. Compute each of the following integrals. Draw the graph of f in each case.

    (a) $\int_0^2 f(x)\,dx$ where $f(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1,\\ 2-x & \text{if } 1 \le x \le 2.\end{cases}$

    (b) $\int_0^1 f(x)\,dx$ where $f(x) = \begin{cases} x & \text{if } 0 \le x \le c,\\ c\dfrac{1-x}{1-c} & \text{if } c \le x \le 1;\end{cases}$

    c is a fixed real number, $0 < c < 1$.
23. Find a quadratic polynomial P for which $P(0) = P(1) = 0$ and $\int_0^1 P(x)\,dx = 1$.
24. Find a cubic polynomial P for which $P(0) = P(-2) = 0$, $P(1) = 15$, and $3\int_{-2}^0 P(x)\,dx = 4$.

Optional exercises

25. Let f be a function whose domain contains $-x$ whenever it contains x. We say that f is an *even* function if $f(-x) = f(x)$ and an *odd* function if $f(-x) = -f(x)$ for all x in the domain of f. If f is integrable on $[0, b]$, prove that

   (a) $\int_{-b}^{b} f(x)\,dx = 2\int_{0}^{b} f(x)\,dx \quad$ if f is even;

   (b) $\int_{-b}^{b} f(x)\,dx = 0 \quad$ if f is odd.

26. Use Theorems 1.18 and 1.19 to derive the formula

$$\int_a^b f(x)\,dx = (b-a)\int_0^1 f[a + (b-a)x]\,dx\,.$$

27. Theorems 1.18 and 1.19 suggest a common generalization for the integral $\int_a^b f(Ax + B)\,dx$. Guess the formula suggested and prove it with the help of Theorems 1.18 and 1.19. Discuss also the case $A = 0$.
28. Use Theorems 1.18 and 1.19 to derive the formula

$$\int_a^b f(c - x)\,dx = \int_{c-b}^{c-a} f(x)\,dx\,.$$

1.27 Proofs of the basic properties of the integral

This section contains proofs of the basic properties of the integral listed in Theorems 1.16 through 1.20 in Section 1.24. We make repeated use of the fact that every function f which is bounded on an interval $[a, b]$ has a lower integral $\underline{I}(f)$ and an upper integral $\bar{I}(f)$ given by

$$\underline{I}(f) = \sup\left\{\int_a^b s \;\middle|\; s \le f\right\}, \qquad \bar{I}(g) = \inf\left\{\int_a^b t \;\middle|\; f \le t\right\},$$

where s and t denote arbitrary step functions below and above f, respectively. We know, by Theorem 1.9, that f is integrable if and only if $\underline{I}(f) = \bar{I}(f)$, in which case the value of the integral of f is the common value of the upper and lower integrals.

Proof of the Linearity Property (*Theorem* 1.16). We decompose the linearity property into two parts:

$$\text{(A)} \qquad \int_a^b (f + g) = \int_a^b f + \int_a^b g\,,$$

$$\text{(B)} \qquad \int_a^b cf = c\int_a^b f\,.$$

To prove (A), let $I(f) = \int_a^b f$ and let $I(g) = \int_a^b g$. We shall prove that $\underline{I}(f + g) = \bar{I}(f + g) = I(f) + I(g)$.

Let s_1 and s_2 denote arbitrary step functions below f and g, respectively. Since f and g are integrable, we have

$$I(f) = \sup\left\{\int_a^b s_1 \;\middle|\; s_1 \le f\right\}, \qquad I(g) = \sup\left\{\int_a^b s_2 \;\middle|\; s_2 \le g\right\}.$$

Observe also that the graph of f is made up of disconnected line segments. There are points on the graph of f where a small change in x produces a sudden jump in the value of the function. Note, however, that the corresponding indefinite integral does not exhibit this behavior. A small change in x produces only a small change in $A(x)$. That is why the graph of A is not disconnected. This illustrates a general property of indefinite integrals known as *continuity*. In the next chapter we shall discuss the concept of continuity in detail and prove that the indefinite integral is always a continuous function.

2.19 Exercises

Evaluate the integrals in Exercises 1 through 16.

1. $\int_0^x (1 + t + t^2)\, dt.$
2. $\int_0^{2y} (1 + t + t^2)\, dt.$
3. $\int_{-1}^{2x} (1 + t + t^2)\, dt.$
4. $\int_1^{1-x} (1 - 2t + 3t^2)\, dt.$
5. $\int_{-2}^x t^2(t^2 + 1)\, dt.$
6. $\int_x^{x^2} (t^2 + 1)^2\, dt.$
7. $\int_1^x (t^{1/2} + 1)\, dt, \qquad x > 0.$
8. $\int_x^{x^2} (t^{1/2} + t^{1/4})\, dt, \qquad x > 0.$
9. $\int_{-\pi}^x \cos t\, dt.$
10. $\int_0^{x^2} (\frac{1}{2} + \cos t)\, dt.$
11. $\int_x^{x^2} (\frac{1}{2} - \sin t)\, dt.$
12. $\int_0^x (u^2 + \sin 3u)\, du.$
13. $\int_x^{x^2} (v^2 + \sin 3v)\, dv.$
14. $\int_0^y (\sin^2 x + x)\, dx.$
15. $\int_0^x \left(\sin 2w + \cos \frac{w}{2}\right) dw.$
16. $\int_{-\pi}^x (\frac{1}{2} + \cos t)^2\, dt.$

17. Find all real values of x such that

$$\int_0^x (t^3 - t)\, dt = \frac{1}{3}\int_{\sqrt{2}}^x (t - t^3)\, dt\,.$$

Draw a suitable figure and interpret the equation geometrically.

18. Let $f(x) = x - [x] - \frac{1}{2}$ if x is not an integer, and let $f(x) = 0$ if x is an integer. (As usual, $[x]$ denotes the greatest integer $\le x$.) Define a new function P as follows:

$$P(x) = \int_0^x f(t)\, dt \qquad \text{for every real } x\,.$$

(a) Draw the graph of f over the interval $[-3, 3]$ and prove that f is periodic with period 1: $f(x + 1) = f(x)$ for all x.
(b) Prove that $P(x) = \frac{1}{2}(x^2 - x)$, if $0 \le x \le 1$ and that P is periodic with period 1.
(c) Express $P(x)$ in terms of $[x]$.
(d) Determine a constant c such that $\int_0^1 (P(t) + c)\, dt = 0$.
(e) For the constant c of part (d), let $Q(x) = \int_0^x (P(t) + c)\, dt$. Prove that Q is periodic with period 1 and that

$$Q(x) = \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12} \qquad \text{if } \ 0 \le x \le 1\,.$$

19. Given an odd function f, defined everywhere, periodic with period 2, and integrable on every interval. Let $g(x) = \int_0^x f(t)\,dt$.
 (a) Prove that $g(2n) = 0$ for every integer n.
 (b) Prove that g is even and periodic with period 2.
20. Given an even function f, defined everywhere, periodic with period 2, and integrable on every interval. Let $g(x) = \int_0^x f(t)\,dt$, and let $A = g(1)$.
 (a) Prove that g is odd and that $g(x + 2) - g(x) = g(2)$.
 (b) Compute $g(2)$ and $g(5)$ in terms of A.
 (c) For what value of A will g be periodic with period 2?
21. Given two functions f and g, integrable on every interval and having the following properties: f is odd, g is even, $f(5) = 7$, $f(0) = 0$, $g(x) = f(x + 5)$, $f(x) = \int_0^x g(t)\,dt$ for all x. Prove that (a) $f(x - 5) = -g(x)$ for all x; (b) $\int_0^5 f(t)\,dt = 7$; (c) $\int_0^x f(t)\,dt = g(0) - g(x)$.

3.20 Exercises

1. Use Theorem 3.16 to establish the following inequalities:

$$\frac{1}{10\sqrt{2}} \le \int_0^1 \frac{x^9}{\sqrt{1+x}}\,dx \le \frac{1}{10}.$$

2. Note that $\sqrt{1-x^2} = (1-x^2)/\sqrt{1-x^2}$ and use Theorem 3.16 to obtain the inequalities

$$\frac{11}{24} \le \int_0^{1/2} \sqrt{1-x^2}\,dx \le \frac{11}{24}\sqrt{\frac{4}{3}}.$$

3. Use the identity $1 + x^6 = (1 + x^2)(1 - x^2 + x^4)$ and Theorem 3.16 to prove that for $a > 0$, we have

$$\frac{1}{1+a^6}\left(a - \frac{a^3}{3} + \frac{a^5}{5}\right) \le \int_0^a \frac{dx}{1+x^2} \le a - \frac{a^3}{3} + \frac{a^5}{5}.$$

Take $a = 1/10$ and calculate the value of the integral rounded off to six decimal places.

4. One of the following two statements is incorrect. Explain why it is wrong.
(a) The integral $\int_{2\pi}^{4\pi} (\sin t)/t\,dt > 0$ because $\int_{2\pi}^{3\pi} (\sin t)/t\,dt > \int_{3\pi}^{4\pi} |\sin t|/t\,dt$.
(b) The integral $\int_{2\pi}^{4\pi} (\sin t)/t\,dt = 0$ because, by Theorem 3.16, for some c between 2π and 4π we have

$$\int_{2\pi}^{4\pi} \frac{\sin t}{t}\,dt = \frac{1}{c}\int_{2\pi}^{4\pi} \sin t\,dt = \frac{\cos(2\pi) - \cos(4\pi)}{c} = 0.$$

5. If n is a positive integer, use Theorem 3.16 to show that

$$\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin(t^2)\,dt = \frac{(-1)^n}{c}, \qquad \text{where } \sqrt{n\pi} \le c \le \sqrt{(n+1)\pi}.$$

6. Assume f is continuous on $[a, b]$. If $\int_a^b f(x)\,dx = 0$, prove that $f(c) = 0$ for at least one c in $[a, b]$.
7. Assume that f is integrable and nonnegative on $[a, b]$. If $\int_a^b f(x)\,dx = 0$, prove that $f(x) = 0$ at each point of continuity of f. [*Hint:* If $f(c) > 0$ at a point of continuity c, there is an interval about c in which $f(x) > \frac{1}{2}f(c)$.]
8. Assume f is continuous on $[a, b]$. Assume also that $\int_a^b f(x)g(x)\,dx = 0$ for every function g that is continuous on $[a, b]$. Prove that $f(x) = 0$ for all x in $[a, b]$.

the minimum is b until the point reaches a certain special position, above which the minimum is $<b$. The exact location of this special position will now be determined.

First of all, we observe that the point (x, y) that minimizes d also minimizes d^2. (This observation enables us to avoid differentiation of square roots.) At this stage, we may express d^2 in terms of x alone or else in terms of y alone. We shall express d^2 in terms of y and leave it as an exercise for the reader to carry out the calculations when d^2 is expressed in terms of x.

Therefore the function f to be minimized is given by the formula

$$f(y) = d^2 = 4y + (y - b)^2 .$$

Although $f(y)$ is defined for all real y, the nature of the problem requires that we seek the minimum only among those $y \geq 0$. The derivative, given by $f'(y) = 4 + 2(y - b)$, is zero only when $y = b - 2$. When $b < 2$, this leads to a negative critical point y which is excluded by the restriction $y \geq 0$. In other words, if $b < 2$, the minimum does not occur at a critical point. In fact, when $b < 2$, we see that $f'(y) > 0$ when $y \geq 0$, and hence f is strictly increasing for $y \geq 0$. Therefore the absolute minimum occurs at the endpoint $y = 0$. The corresponding minimum d is $\sqrt{b^2} = |b|$.

If $b \geq 2$, there is a legitimate critical point at $y = b - 2$. Since $f''(y) = 2$ for all y, the derivative f' is increasing, and hence the *absolute minimum* of f occurs at this critical point. The minimum d is $\sqrt{4(b - 2) + 4} = 2\sqrt{b - 1}$. Thus we have shown that the minimum distance is $|b|$ if $b < 2$ and is $2\sqrt{b - 1}$ if $b \geq 2$. (The value $b = 2$ is the special value referred to above.)

4.21 Exercises

1. Prove that among all rectangles of a given area, the square has the smallest perimeter.
2. A farmer has L feet of fencing to enclose a rectangular pasture adjacent to a long stone wall. What dimensions give the maximum area of the pasture?
3. A farmer wishes to enclose a rectangular pasture of area A adjacent to a long stone wall. What dimensions require the least amount of fencing?
4. Given $S > 0$. Prove that among all positive numbers x and y with $x + y = S$, the sum $x^2 + y^2$ is smallest when $x = y$.
5. Given $R > 0$. Prove that among all positive numbers x and y with $x^2 + y^2 = R$, the sum $x + y$ is largest when $x = y$.
6. Each edge of a square has length L. Prove that among all squares inscribed in the given square, the one of minimum area has edges of length $\frac{1}{2}L\sqrt{2}$.
7. Each edge of a square has length L, Find the size of the square of largest area that can be circumscribed about the given square.
8. Prove that among all rectangles that can be inscribed in a given circle, the square has the largest area.
9. Prove that among all rectangles of a given area, the square has the smallest circumscribed circle.
10. Given a sphere of radius R. Find the radius r and altitude h of the right circular cylinder with largest lateral surface area $2\pi rh$ that can be inscribed in the sphere.
11. Among all right circular cylinders of given lateral surface area, prove that the smallest circumscribed sphere has radius $\sqrt{2}$ times that of the cylinder.

12. Given a right circular cone with radius R and altitude H. Find the radius and altitude of the right circular cylinder of largest lateral surface area that can be inscribed in the cone.
13. Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a right circular cone of radius R and altitude H.
14. Given a sphere of radius R. Compute, in terms of R, the radius r and the altitude h of the right circular cone of maximum volume that can be inscribed in this sphere.
15. Find the rectangle of largest area that can be inscribed in a semicircle, the lower base being on the diameter.
16. Find the trapezoid of largest area that can be inscribed in a semicircle, the lower base being on the diameter.
17. An open box is made from a rectangular piece of material by removing equal squares at each corner and turning up the sides. Find the dimensions of the box of largest volume that can be made in this manner if the material has sides (a) 10 and 10; (b) 12 and 18.
18. If a and b are the legs of a right triangle whose hypotenuse is 1, find the largest value of $2a + b$.
19. A truck is to be driven 300 miles on a freeway at a constant speed of x miles per hour. Speed laws require $30 \leq x \leq 60$. Assume that fuel costs 30 cents per gallon and is consumed at the rate of $2 + x^2/600$ gallons per hour. If the driver's wages are D dollars per hour and if he obeys all speed laws, find the most economical speed and the cost of the trip if (a) $D = 0$, (b) $D = 1$, (c) $D = 2$, (d) $D = 3$, (e) $D = 4$.
20. A cylinder is obtained by revolving a rectangle about the x-axis, the base of the rectangle lying on the x-axis and the entire rectangle lying in the region between the curve $y = x/(x^2 + 1)$ and the x-axis. Find the maximum possible volume of the cylinder.
21. The lower right-hand corner of a page is folded over so as to reach the leftmost edge. (See Figure 4.17.) If the width of the page is six inches, find the minimum length of the crease. What angle will this minimal crease make with the rightmost edge of the page? Assume the page is long enough to prevent the crease reaching the top of the page.

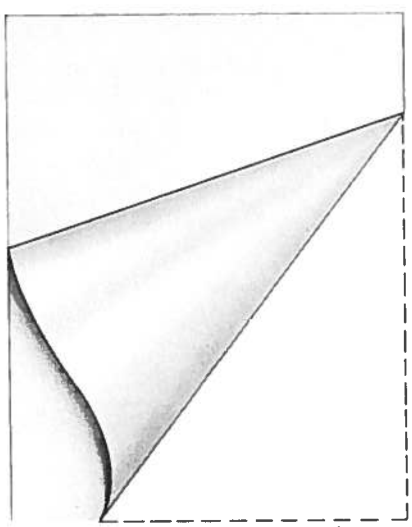

FIGURE 4.17 Exercise 21.

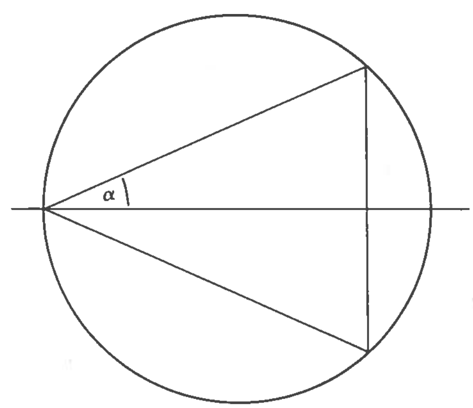

FIGURE 4.18 Exercise 22.

22. (a) An isosceles triangle is inscribed in a circle of radius r as shown in Figure 4.18. If the angle 2α at the apex is restricted to lie between 0 and $\frac{1}{2}\pi$, find the largest value and the smallest value of the perimeter of the triangle. Give full details of your reasoning.

(b) What is the radius of the smallest circular disk large enough to cover *every* isosceles triangle of a given perimeter L? Give full details of your reasoning.

23. A window is to be made in the form of a rectangle surmounted by a semicircle with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass, and the semicircular portion is to be of a colored glass admitting only half as much light per square foot as the clear glass. The total perimeter of the window frame is to be a fixed length P. Find, in terms of P, the dimensions of the window which will admit the most light.
24. A log 12 feet long has the shape of a frustum of a right circular cone with diameters 4 feet and $(4 + h)$ feet at its ends, where $h \geq 0$. Determine, as a function of h, the volume of the largest right circular cylinder that can be cut from the log, if its axis coincides with that of the log.
25. Given n real numbers $a_1, \ldots, a_n$. Prove that the sum $\sum_{k=1}^{n} (x - a_k)^2$ is smallest when x is the arithmetic mean of $a_1, \ldots, a_n$.
26. If $x > 0$, let $f(x) = 5x^2 + Ax^{-5}$, where A is a positive constant. Find the smallest A such that $f(x) \geq 24$ for all $x > 0$.
27. For each real t, let $f(x) = -\frac{1}{3}x^3 + t^2x$, and let $m(t)$ denote the minimum of $f(x)$ over the interval $0 \leq x \leq 1$. Determine the value of $m(t)$ for each t in the interval $-1 \leq t \leq 1$. Remember that for some values of t the minimum of $f(x)$ may occur at the endpoints of the interval $0 \leq x \leq 1$.
28. A number x is known to lie in an interval $a \leq x \leq b$, where $a > 0$. We wish to approximate x by another number t in $[a, b]$ so that the relative error, $|t - x|/x$, will be as small as possible. Let $M(t)$ denote the maximum value of $|t - x|/x$ as x varies from a to b. (a) Prove that this maximum occurs at one of the endpoints $x = a$ or $x = b$. (b) Prove that $M(t)$ is smallest when t is the harmonic mean of a and b, that is, when $1/t = \frac{1}{2}(1/a + 1/b)$.

*4.22 Partial derivatives

This section explains the concept of partial derivative and introduces the reader to some notation and terminology. We shall not make use of the results of this section anywhere else in Volume I, so this material may be omitted or postponed without loss in continuity.

In Chapter 1, a function was defined to be a correspondence which associates with each object in a set X one and only one object in another set Y; the set X is referred to as the *domain* of the function. Up to now, we have dealt with functions having a domain consisting of points on the x-axis. Such functions are usually called *functions of one real variable*. It is not difficult to extend many of the ideas of calculus to functions of two or more real variables.

By a *real-valued function of two real variables* we mean one whose domain X is a set of points in the xy-plane. If f denotes such a function, its value at a point (x, y) is a real number, written $f(x, y)$. It is easy to imagine how such a function might arise in a physical problem. For example, suppose a flat metal plate in the shape of a circular disk of radius 4 centimeters is placed on the xy-plane, with the center of the disk at the origin and with the disk heated in such a way that its temperature at each point (x, y) is $16 - x^2 - y^2$ degrees centigrade. If we denote the temperature at (x, y) by $f(x, y)$, then f is a function of two variables defined by the equation

$$f(x, y) = 16 - x^2 - y^2. \tag{4.27}$$

The domain of this function is the set of all points (x, y) whose distance from the origin does not exceed 4. The theorem of Pythagoras tells us that all points (x, y) at a distance

For the example in (4.30), we obtain

$$\begin{aligned}
f_{1,1}(x, y) &= -y^4 \cos xy\,, \\
f_{1,2}(x, y) &= \cos y - xy^3 \cos xy - 3y^2 \sin xy\,, \\
f_{2,1}(x, y) &= \cos y - xy^3 \cos xy - y^2 \sin xy - 2y^2 \sin xy = f_{1,2}(x, y)\,, \\
f_{2,2}(x, y) &= -x \sin y - x^2y^2 \cos xy - 2xy \sin xy - 2xy \sin xy + 2 \cos xy \\
&= -x \sin y - x^2y^2 \cos xy - 4xy \sin xy + 2 \cos xy\,.
\end{aligned}$$

A more detailed study of partial derivatives will be undertaken in Volume II.

*4.23 Exercises

In Exercises 1 through 8, compute all first- and second-order partial derivatives. In each case verify that the mixed partial derivatives $f_{1,2}(x, y)$ and $f_{2,1}(x, y)$ are equal.

1. $f(x, y) = x^4 + y^4 - 4x^2y^2$.
2. $f(x, y) = x \sin (x + y)$.
3. $f(x, y) = xy + \dfrac{x}{y} \quad (y \neq 0)$.
4. $f(x, y) = \sqrt{x^2 + y^2}$.
5. $f(x, y) = \sin (x^2y^3)$.
6. $f(x, y) = \sin [\cos (2x - 3y)]$.
7. $f(x, y) = \dfrac{x + y}{x - y} \quad (x \neq y)$.
8. $f(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}} \quad (x, y) \neq (0, 0)$.
9. Show that $x(\partial z/\partial x) + y(\partial z/\partial y) = 2z$ if (a) $z = (x - 2y)^2$, (b) $z = (x^4 + y^4)^{1/2}$.
10. If $f(x, y) = xy/(x^2 + y^2)^2$ for $(x, y) \neq (0, 0)$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

16. (a) If $I_n(x) = \int_0^x t^n(t^2 + a^2)^{-1/2}\,dt$, use integration by parts to show that

$$nI_n(x) = x^{n-1}\sqrt{x^2 + a^2} - (n - 1)a^2 I_{n-2}(x) \qquad \text{if} \quad n \geq 2\,.$$

(b) Use part (a) to show that $\int_0^2 x^5(x^2 + 5)^{-1/2}\,dx = 168/5 - 40\sqrt{5}/3$.

17. Evaluate the integral $\int_{-1}^3 t^3(4 + t^3)^{-1/2}\,dt$, given that $\int_{-1}^3 (4 + t^3)^{1/2}\,dt = 11.35$. Leave the answer in terms of $\sqrt{3}$ and $\sqrt{31}$.

18. Use integration by parts to derive the formula

$$\int \frac{\sin^{n+1} x}{\cos^{m+1} x}\,dx = \frac{1}{m}\frac{\sin^n x}{\cos^m x} - \frac{n}{m}\int \frac{\sin^{n-1} x}{\cos^{m-1} x}\,dx\,.$$

Apply the formula to integrate $\int \tan^2 x\,dx$ and $\int \tan^4 x\,dx$.

19. Use integration by parts to derive the formula

$$\int \frac{\cos^{m+1} x}{\sin^{n+1} x}\,dx = -\frac{1}{n}\frac{\cos^m x}{\sin^n x} - \frac{m}{n}\int \frac{\cos^{m-1} x}{\sin^{n-1} x}\,dx\,.$$

Apply the formula to integrate $\int \cot^2 x\,dx$ and $\int \cot^4 x\,dx$.

20. (a) Find an integer n such that $n\int_0^1 xf''(2x)\,dx = \int_0^2 tf''(t)\,dt$.
(b) Compute $\int_0^1 xf''(2x)\,dx$, given that $f(0) = 1$, $f(2) = 3$, and $f'(2) = 5$.

21. (a) If ϕ'' is continuous and nonzero on $[a, b]$, and if there is a constant $m > 0$ such that $\phi'(t) \geq m$ for all t in $[a, b]$, use Theorem 5.5 to prove that

$$\left|\int_a^b \sin \phi(t)\,dt\right| \leq \frac{4}{m}\,.$$

[*Hint:* Multiply and divide the integrand by $\phi'(t)$.]

(b) If $a > 0$, show that $\left|\int_a^x \sin (t^2)\,dt\right| \leq 2/a$ for all $x > a$.

*5.11 Miscellaneous review exercises

1. Let f be a polynomial with $f(0) = 1$ and let $g(x) = x^n f(x)$. Compute $g(0), g'(0), \ldots, g^{(n)}(0)$.
2. Find a polynomial P of degree ≤ 5 with $P(0) = 1, P(1) = 2, P'(0) = P''(0) = P'(1) = P''(1) = 0$.
3. If $f(x) = \cos x$ and $g(x) = \sin x$, prove that

$$f^{(n)}(x) = \cos (x + \tfrac{1}{2}n\pi) \qquad \text{and} \qquad g^{(n)}(x) = \sin (x + \tfrac{1}{2}n\pi).$$

4. If $h(x) = f(x)g(x)$, prove that the nth derivative of h is given by the formula

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x)\,,$$

where $\binom{n}{k}$ denotes the binomial coefficient. This is called *Leibniz's formula*.

5. Given two functions f and g whose derivatives f' and g' satisfy the equations

$$f'(x) = g(x)\,, \qquad g'(x) = -f(x)\,, \qquad f(0) = 0\,, \qquad g(0) = 1\,, \tag{5.30}$$

for every x in some open interval J containing 0. For example, these equations are satisfied when $f(x) = \sin x$ and $g(x) = \cos x$.

(a) Prove that $f^2(x) + g^2(x) = 1$ for every x in J.
(b) Let F and G be another pair of functions satisfying (5.30). Prove that $F(x) = f(x)$ and $G(x) = g(x)$ for every x in J. [*Hint:* Consider $h(x) = [F(x) - f(x)]^2 + [G(x) - g(x)]^2$.]
(c) What more can you say about functions f and g satisfying (5.30)?

6. A function f, defined for all positive real numbers, satisfies the equation $f(x^2) = x^3$ for every $x > 0$. Determine $f'(4)$.
7. A function g, defined for all positive real numbers, satisfies the following two conditions: $g(1) = 1$ and $g'(x^2) = x^3$ for all $x > 0$. Compute $g(4)$.
8. Show that

$$\int_0^x \frac{\sin t}{t+1}\,dt \geq 0 \qquad \text{for all} \quad x \geq 0\,.$$

9. Let C_1 and C_2 be two curves passing through the origin as indicated in Figure 5.2. A curve C is said to "bisect in area" the region between C_1 and C_2 if, for each point P of C, the two shaded regions A and B shown in the figure have equal areas. Determine the upper curve C_2, given that the bisecting curve C has the equation $y = x^2$ and that the lower curve C_1 has the equation $y = \frac{1}{2}x^2$.

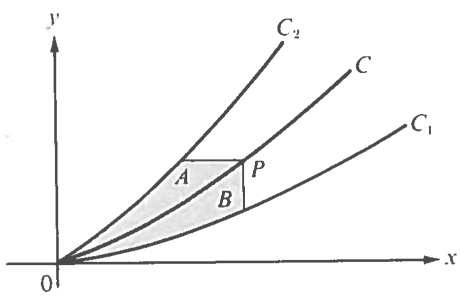

FIGURE 5.2 Exercise 9.

10. A function f is defined for all x as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

Let $Q(h) = f(h)/h$ if $h \neq 0$. (a) Prove that $Q(h) \to 0$ as $h \to 0$. (b) Prove that f has a derivative at 0, and compute $f'(0)$.

In Exercises 11 through 20, evaluate the given integrals. Try to simplify the calculations by using the method of substitution and/or integration by parts whenever possible.

11. $\int (2 + 3x) \sin 5x\, dx$.
12. $\int x\sqrt{1 + x^2}\, dx$.
13. $\int_{-2}^{1} x(x^2 - 1)^9\, dx$.
14. $\int_0^1 \frac{2x + 3}{(6x + 7)^3}\, dx$.
15. $\int x^4(1 + x^5)^5\, dx$.
16. $\int_0^1 x^4(1 - x)^{20}\, dx$.
17. $\int_1^2 x^{-2} \sin \frac{1}{x}\, dx$.
18. $\int \sin \sqrt[4]{x - 1}\, dx$.
19. $\int x \sin x^2 \cos x^2\, dx$.
20. $\int \sqrt{1 + 3\cos^2 x}\, \sin 2x\, dx$.

21. Show that the value of the integral $\int_0^2 375x^5(x^2+1)^{-4}\,dx$ is 2^n for some integer n.
22. Determine a pair of numbers a and b for which $\int_0^1 (ax+b)(x^2+3x+2)^{-2}\,dx = 3/2$.
23. Let $I_n = \int_0^1 (1-x^2)^n\,dx$. Show that $(2n+1)I_n = 2n\,I_{n-1}$, then use this relation to compute I_2, I_3, I_4, and I_5.
24. Let $F(m, n) = \int_0^x t^m(1+t)^n\,dt$, $m > 0$, $n > 0$. Show that

$$(m+1)F(m,n) + nF(m+1, n-1) = x^{m+1}(1+x)^n.$$

Use this to evaluate $F(10, 2)$.
25. Let $f(n) = \int_0^{\pi/4} \tan^n x\,dx$ where $n \geq 1$. Show that
(a) $f(n+1) < f(n)$.

(b) $f(n) + f(n-2) = \dfrac{1}{n-1}$ if $n > 2$.

(c) $\dfrac{1}{n+1} < 2f(n) < \dfrac{1}{n-1}$ if $n > 2$.

26. Compute $f(0)$, given that $f(\pi) = 2$ and that $\int_0^\pi [f(x) + f''(x)] \sin x\,dx = 5$.
27. Let A denote the value of the integral

$$\int_0^\pi \frac{\cos x}{(x+2)^2}\,dx\,.$$

Compute the following integral in terms of A:

$$\int_0^{\pi/2} \frac{\sin x \cos x}{x+1}\,dx\,.$$

The formulas in Exercises 28 through 33 appear in integral tables. Verify each of these formulas by any method.

28. $$\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int \frac{dx}{x\sqrt{a+bx}} + C.$$

29. $$\int x^n\sqrt{ax+b}\,dx = \frac{2}{a(2n+3)}\left(x^n(ax+b)^{3/2} - nb\int x^{n-1}\sqrt{ax+b}\,dx\right) + C \quad (n \neq -\tfrac{3}{2}).$$

30. $$\int \frac{x^m}{\sqrt{a+bx}}\,dx = \frac{2}{(2m+1)b}\left(x^m\sqrt{a+bx} - ma\int \frac{x^{m-1}}{\sqrt{a+bx}}\,dx\right) + C \quad (m \neq -\tfrac{1}{2}).$$

31. $$\int \frac{dx}{x^n\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{(n-1)bx^{n-1}} - \frac{(2n-3)a}{(2n-2)b}\int \frac{dx}{x^{n-1}\sqrt{ax+b}} + C \quad (n \neq 1).$$

32. $$\int \frac{\cos^m x}{\sin^n x}\,dx = \frac{\cos^{m-1} x}{(m-n)\sin^{n-1} x} + \frac{m-1}{m-n}\int \frac{\cos^{m-2} x}{\sin^n x}\,dx + C \quad (m \neq n).$$

33. $$\int \frac{\cos^m x}{\sin^n x}\,dx = -\frac{\cos^{m+1} x}{(n-1)\sin^{n-1} x} - \frac{m-n+2}{n-1}\int \frac{\cos^m x}{\sin^{n-2} x}\,dx + C \quad (n \neq 1).$$

34. (a) Find a polynomial $P(x)$ such that $P'(x) - 3P(x) = 4 - 5x + 3x^2$. Prove that there is only one solution.

(b) If $Q(x)$ is a given polynomial, prove that there is one and only one polynomial $P(x)$ such that $P'(x) - 3P(x) = Q(x)$.

35. A sequence of polynomials (called the *Bernoulli polynomials*) is defined inductively as follows:

$$P_0(x) = 1; \qquad P_n'(x) = nP_{n-1}(x) \qquad \text{and} \qquad \int_0^1 P_n(x)\,dx = 0 \qquad \text{if} \quad n \geq 1.$$

(a) Determine explicit formulas for $P_1(x), P_2(x), \ldots, P_5(x)$.
(b) Prove, by induction, that $P_n(x)$ is a polynomial in x of degree n, the term of highest degree being x^n.
(c) Prove that $P_n(0) = P_n(1)$ if $n \geq 2$.
(d) Prove that $P_n(x + 1) - P_n(x) = nx^{n-1}$ if $n \geq 1$.
(e) Prove that for $n \geq 2$ we have

$$\sum_{r=1}^{k-1} r^n = \int_0^k P_n(x)\,dx = \frac{P_{n+1}(k) - P_{n+1}(0)}{n + 1}.$$

(f) Prove that $P_n(1 - x) = (-1)^n P_n(x)$ if $n \geq 1$.
(g) Prove that $P_{2n+1}(0) = 0$ and $P_{2n-1}(\frac{1}{2}) = 0$ if $n \geq 1$.

36. Assume that $|f''(x)| \leq m$ for each x in the interval $[0, a]$, and assume that f takes on its largest value at an interior point of this interval. Show that $|f'(0)| + |f'(a)| \leq am$. You may assume that f'' is continuous in $[0, a]$.

39. $\displaystyle\int \frac{dx}{\sqrt{x^2 + x}}$.

40. $\displaystyle\int \frac{\sqrt{2 - x - x^2}}{x^2}\, dx$.

[*Hint:* In Exercise 40, multiply numerator and denominator by $\sqrt{2 - x - x^2}$.]

6.26 Miscellaneous review exercises

1. Let $f(x) = \int_1^x (\log t)/(t + 1)\, dt$ if $x > 0$. Compute $f(x) + f(1/x)$. As a check, you should obtain $f(2) + f(\frac{1}{2}) = \frac{1}{2}\log^2 2$.
2. Find a function f, continuous for all x (and not everywhere zero), such that

$$f^2(x) = \int_0^x f(t)\frac{\sin t}{2 + \cos t}\, dt\,.$$

3. Try to evaluate $\int e^x/x\, dx$ by using integration by parts.
4. Integrate $\int_0^{\pi/2} \log(e^{\cos x})\, dx$.
5. A function f is defined by the equation

$$f(x) = \sqrt{\frac{4x + 2}{x(x + 1)(x + 2)}} \qquad \text{if} \quad x > 0\,.$$

(a) Find the slope of the graph of f at the point for which $x = 1$.
(b) The region under the graph and above the interval $[1, 4]$ is rotated about the x-axis, thus generating a solid of revolution. Write an integral for the volume of this solid. Compute this integral and show that its value is $\pi \log(25/8)$.

6. A function F is defined by the following indefinite integral:

$$F(x) = \int_1^x \frac{e^t}{t}\, dt \qquad \text{if} \quad x > 0\,.$$

(a) For what values of x is it true that $\log x \le F(x)$?
(b) Prove that $\int_1^x e^t/(t + a)\, dt = e^{-a}[F(x + a) - F(1 + a)]$.
(c) In a similar way, express the following integrals in terms of F:

$$\int_1^x \frac{e^{at}}{t}\, dt, \qquad \int_1^x \frac{e^t}{t^2}\, dt, \qquad \int_1^x e^{1/t}\, dt\,.$$

7. In each case, give an example of a continuous function f satisfying the conditions stated for all real x, or else explain why there is no such function:
(a) $\int_0^x f(t)\, dt = e^x$.
(b) $\int_0^{x^2} f(t)\, dt = 1 - 2^{x^2}$. [$2^{x^2}$ means $2^{(x^2)}$.]
(c) $\int_0^x f(t)\, dt = f^2(x) - 1$.
8. If $f(x + y) = f(x)f(y)$ for all x and y and if $f(x) = 1 + xg(x)$, where $g(x) \to 1$ as $x \to 0$, prove that (a) $f'(x)$ exists for every x, and (b) $f(x) = e^x$.
9. Given a function g which has a derivative $g'(x)$ for every real x and which satisfies the following equations:

$$g'(0) = 2 \qquad \text{and} \qquad g(x + y) = e^y g(x) + e^x g(y) \qquad \text{for all } x \text{ and } y\,.$$

(a) Show that $g(2x) = 2e^x g(x)$ and find a similar formula for $g(3x)$.
(b) Generalize (a) by finding a formula relating $g(nx)$ to $g(x)$, valid for every positive integer n. Prove your result by induction.

(c) Show that $g(0) = 0$ and find the limit of $g(h)/h$ as $h \to 0$.
(d) There is a constant C such that $g'(x) = g(x) + Ce^x$ for all x. Prove this statement and find the value of C. [*Hint:* Use the definition of the derivative $g'(x)$.]

10. A periodic function with period a satisfies $f(x + a) = f(x)$ for all x in its domain. What can you conclude about a function which has a derivative everywhere and satisfies an equation of the form
$$f(x + a) = bf(x)$$
for all x, where a and b are positive constants?

11. Use logarithmic differentiation to derive the formulas for differentiation of products and quotients from the corresponding formulas for sums and differences.

12. Let $A = \int_0^1 e^t/(t + 1)\, dt$. Express the values of the following integrals in terms of A:

(a) $\displaystyle\int_{a-1}^{a} \frac{e^{-t}}{t - a - 1}\, dt.$

(b) $\displaystyle\int_0^1 \frac{te^{t^2}}{t^2 + 1}\, dt.$

(c) $\displaystyle\int_0^1 \frac{e^t}{(t + 1)^2}\, dt.$

(d) $\displaystyle\int_0^1 e^t \log(1 + t)\, dt.$

13. Let $p(x) = c_0 + c_1x + c_2x^2$ and let $f(x) = e^x p(x)$.
(a) Show that $f^{(n)}(0)$, the nth derivative of f at 0, is $c_0 + nc_1 + n(n - 1)c_2$.
(b) Solve the problem when p is a polynomial of degree 3.
(c) Generalize to a polynomial of degree m.

14. Let $f(x) = x \sin ax$. Show that $f^{(2n)}(x) = (-1)^n(a^{2n}x \sin ax - 2na^{2n-1} \cos ax)$.

15. Prove that
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k + m + 1} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{k + n + 1}.$$
[*Hint:* $1/(k + m + 1) = \int_0^1 t^{k+m}\, dt$.]

16. Let $F(x) = \int_0^x f(t)\, dt$. Determine a formula (or formulas) for computing $F(x)$ for all real x if f is defined as follows:

(a) $f(t) = (t + |t|)^2$.

(b) $f(t) = \begin{cases} 1 - t^2 & \text{if } |t| \le 1, \\ 1 - |t| & \text{if } |t| > 1. \end{cases}$

(c) $f(t) = e^{-|t|}$.

(d) $f(t) =$ the maximum of 1 and t^2.

17. A solid of revolution is generated by rotating the graph of a continuous function f around the interval $[0, a]$ on the x-axis. If, for every $a > 0$, the volume is $a^2 + a$, find the function f.

18. Let $f(x) = e^{-2x}$ for all x. Denote by $S(t)$ the ordinate set of f over the interval $[0, t]$, where $t > 0$. Let $A(t)$ be the area of $S(t)$, $V(t)$ the volume of the solid obtained by rotating $S(t)$ about the x-axis, and $W(t)$ the volume of the solid obtained by rotating $S(t)$ about the y-axis. Compute the following: (a) $A(t)$; (b) $V(t)$; (c) $W(t)$; (d) $\lim_{t\to 0} V(t)/A(t)$.

19. Let c be the number such that $\sinh c = \frac{3}{4}$. (Do not attempt to compute c.) In each case find all those x (if any exist) satisfying the given equation. Express your answers in terms of $\log 2$ and $\log 3$.
(a) $\log(e^x + \sqrt{e^{2x} + 1}) = c$.
(b) $\log(e^x - \sqrt{e^{2x} - 1}) = c$.

20. Determine whether each of the following statements is true or false. Prove each true statement.

(a) $2^{\log 5} = 5^{\log 2}$.

(b) $\log_2 5 = \dfrac{\log_3 5}{\log_2 3}$.

(c) $\displaystyle\sum_{k=1}^{n} k^{-1/2} < 2\sqrt{n}$ for every $n \ge 1$.

(d) $1 + \sinh x \le \cosh x$ for every x.

In Exercises 21 through 24, establish each inequality by examining the sign of the derivative of an appropriate function.

21. $\frac{2}{\pi}x < \sin x < x \quad$ if $\quad 0 < x < \frac{\pi}{2}$.

22. $\frac{1}{x + \frac{1}{2}} < \log\left(1 + \frac{1}{x}\right) < \frac{1}{x} \quad$ if $\quad x > 0$.

23. $x - \frac{x^3}{6} < \sin x < x \quad$ if $\quad x > 0$.

24. $(x^b + y^b)^{1/b} < (x^a + y^a)^{1/a} \quad$ if $\quad x > 0, y > 0$, and $\quad 0 < a < b$.

25. Show that
   (a) $\int_0^x e^{-t}\, t\, dt = e^{-x}(e^x - 1 - x)$.
   (b) $\int_0^x e^{-t}t^2\, dt = 2!e^{-x}\left(e^x - 1 - x - \frac{x^2}{2!}\right)$.
   (c) $\int_0^x e^{-t}t^3\, dt = 3!e^{-x}\left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!}\right)$.
   (d) Guess the generalization suggested and prove it by induction.

26. If a, b, a_1, b_1 are given, with $ab \neq 0$, show that there exist constants A, B, C such that

$$\int \frac{a_1 \sin x + b_1 \cos x}{a \sin x + b \cos x}\, dx = Ax + B \log |a \sin x + b \cos x| + C.$$

[*Hint:* Show that A and B exist such that

$$a_1 \sin x + b_1 \cos x = A(a \sin x + b \cos x) + B(a \cos x - b \sin x).]$$

27. In each case, find a function f satisfying the given conditions.
   (a) $f'(x^2) = 1/x \quad$ for $x > 0$, $\quad f(1) = 1$.
   (b) $f'(\sin^2 x) = \cos^2 x \quad$ for all x, $\quad f(1) = 1$.
   (c) $f'(\sin x) = \cos^2 x \quad$ for all x, $\quad f(1) = 1$.
   (d) $f'(\log x) = \begin{cases} 1 & \text{for } 0 < x \le 1, \\ x & \text{for } x > 1, \end{cases} \quad f(0) = 0$.

28. A function, called the *integral logarithm* and denoted by Li, is defined as follows:

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \quad \text{if } \quad x \ge 2.$$

This function occurs in analytic number theory where it is proved that $\mathrm{Li}(x)$ is a very good approximation to the number of primes $\le x$. Derive the following properties of $\mathrm{Li}(x)$:

   (a) $\mathrm{Li}(x) = \frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t} - \frac{2}{\log 2}$.
   (b) $\mathrm{Li}(x) = \frac{x}{\log x} + \sum_{k=1}^{n-1} \frac{k!\, x}{\log^{k+1} x} + n! \int_2^x \frac{dt}{\log^{n+1} t} + C_n$,
   where C_n is a constant (depending on n). Find this constant.
   (c) Show that there is a constant b such that $\int_b^{\log x} e^t/t\, dt = \mathrm{Li}(x)$ and find the value of b.
   (d) Express $\int_c^x e^{2t}/(t - 1)\, dt$ in terms of the integral logarithm, where $c = 1 + \frac{1}{2}\log 2$.

(e) Let $f(x) = e^4 \operatorname{Li}(e^{2x-4}) - e^2 \operatorname{Li}(e^{2x-2})$ if $x > 3$. Show that

$$f'(x) = \frac{e^{2x}}{x^2 - 3x + 2}.$$

29. Let $f(x) = \log |x|$ if $x < 0$. Show that f has an inverse, and denote this inverse by g. What is the domain of g? Find a formula for computing $g(y)$ for each y in the domain of g. Sketch the graph of g.

30. Let $f(x) = \int_0^x (1 + t^3)^{-1/2}\, dt$ if $x \geq 0$. (Do not attempt to evaluate this integral.)
(a) Show that f is strictly increasing on the nonnegative real axis.
(b) Let g denote the inverse of f. Show that the second derivative of g is proportional to g^2 [that is, $g''(y) = cg^2(y)$ for each y in the domain of g] and find the constant of proportionality.

7.17 Exercises

Evaluate the limits in Exercises 1 through 25. The letters a and b denote positive constants.

1. $\lim_{x\to 0} \dfrac{e^{-1/x^2}}{x^{1000}}$.

2. $\lim_{x\to+\infty} \dfrac{\sin(1/x)}{\arctan(1/x)}$.

3. $\lim_{x\to\frac{1}{2}\pi} \dfrac{\tan 3x}{\tan x}$.

4. $\lim_{x\to+\infty} \dfrac{\log(a+be^x)}{\sqrt{a+bx^2}}$.

5. $\lim_{x\to+\infty} x^4\left(\cos\dfrac{1}{x} - 1 + \dfrac{1}{2x^2}\right)$.

6. $\lim_{x\to\pi} \dfrac{\log|\sin x|}{\log|\sin 2x|}$.

7. $\lim_{x\to\frac{1}{2}-} \dfrac{\log(1-2x)}{\tan \pi x}$.

8. $\lim_{x\to+\infty} \dfrac{\cosh(x+1)}{e^x}$.

9. $\lim_{x\to+\infty} \dfrac{a^x}{x^b}$, $\quad a > 1$.

10. $\lim_{x\to\frac{1}{2}\pi} \dfrac{\tan x - 5}{\sec x + 4}$.

11. $\lim_{x\to 0+} \dfrac{1}{\sqrt{x}}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$.

12. $\lim_{x\to+\infty} x^{1/4}\sin(1/\sqrt{x})$.

13. $\lim_{x\to+\infty} (x^2 - \sqrt{x^4 - x^2 + 1})$.

14. $\lim_{x\to 0+}\left[\dfrac{\log x}{(1+x)^2} - \log\left(\dfrac{x}{1+x}\right)\right]$.

15. $\lim_{x\to 1-} (\log x)\log(1-x)$.

16. $\lim_{x\to 0+} x^{(x^x-1)}$.

17. $\lim_{x\to 0+} [x^{(x^x)} - 1]$.

18. $\lim_{x\to 0-} (1-2^x)^{\sin x}$.

19. $\lim_{x\to 0+} x^{1/\log x}$.

20. $\lim_{x\to 0+} (\cot x)^{\sin x}$.

21. $\lim_{x\to\frac{1}{4}\pi} (\tan x)^{\tan 2x}$.

22. $\lim_{x\to 0+} \left(\log\dfrac{1}{x}\right)^x$.

23. $\lim_{x\to 0+} x^{e/(1+\log x)}$.

24. $\lim_{x\to 1} (2-x)^{\tan(\pi x/2)}$.

25. $\lim_{x\to 0}\left(\dfrac{1}{\log(x+\sqrt{1+x^2})} - \dfrac{1}{\log 1(+x)}\right)$.

26. Find c so that

$$\lim_{x\to+\infty}\left(\frac{x+c}{x-c}\right)^x = 4.$$

27. Prove that $(1+x)^c = 1 + cx + o(x)$ as $x \to 0$. Use this to compute the limit of

$$\{(x^4 + x^2)^{1/2} - x^2\} \qquad \text{as } x \to +\infty.$$

28. For a certain value of c, the limit

$$\lim_{x\to+\infty} \{(x^5 + 7x^4 + 2)^c - x\}$$

is finite and nonzero. Determine this c and compute the value of the limit.

29. Let $g(x) = xe^{x^2}$ and let $f(x) = \int_1^x g(t)(t + 1/t)\,dt$. Compute the limit of $f''(x)/g''(x)$ as $x \to +\infty$.
30. Let $g(x) = x^c e^{2x}$ and let $f(x) = \int_0^x e^{2t}(3t^2 + 1)^{1/2}\,dt$. For a certain value of c, the limit of $f'(x)/g'(x)$ as $x \to +\infty$ is finite and nonzero. Determine c and compute the value of the limit.
31. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$, and let $f(0) = 0$.
    (a) Prove that for every $m > 0$, $f(x)/x^m \to 0$ as $x \to 0$.
    (b) Prove that for $x \neq 0$ the nth derivative of f has the form $f^{(n)}(x) = f(x)P(1/x)$, where $P(t)$ is a polynomial in t.
    (c) Prove that $f^{(n)}(0) = 0$ for all $n \geq 1$. This shows that every Taylor polynomial generated by f at 0 is the zero polynomial.
32. An amount of P dollars is deposited in a bank which pays interest at a rate r per year, compounded m times a year. (For example, $r = 0.06$ when the annual rate is 6%.) (a) Prove that the total amount of principal plus interest at the end of n years is $P(1 + r/m)^{mn}$. If r and n are kept fixed, this amount approaches the limit Pe^{rn} as $m \to +\infty$. This motivates the following definition: We say that money grows at an annual rate r when compounded continuously if the amount $f(t)$ after t years is $f(0)e^{rt}$, where t is any nonnegative real number. Approximately how long does it take for a bank account to double in value if it receives interest at an annual rate of 6% compounded (b) continuously? (c) four times a year?

To lay a better foundation for the general theory of power series, we turn next to certain general questions related to the convergence and divergence of arbitrary series. We shall return to the subject of power series in Chapter 11.

10.9 Exercises

Each of the series in Exercises 1 through 10 is a telescoping series, or a geometric series, or some related series whose partial sums may be simplified. In each case, prove that the series converges and has the sum indicated.

1. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$.
2. $\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3$.
3. $\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$.
4. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n} = \frac{3}{2}$.
5. $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}} = 1$.
6. $\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)} = \frac{1}{4}$.
7. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1$.
8. $\sum_{n=1}^{\infty} \frac{2^n + n^2 + n}{2^{n+1}n(n+1)} = 1$.
9. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n+1)}{n(n+1)} = 1$.
10. $\sum_{n=2}^{\infty} \frac{\log[(1+1/n)^n(1+n)]}{(\log n^n)[\log (n+1)^{n+1}]} = \log_2 \sqrt{e}$.

Power series for $\log(1+x)$ and $\arctan x$ were obtained in Section 10.8 by performing various operations on the geometric series. In a similar manner, without attempting to justify the steps, obtain the formulas in Exercises 11 through 19. They are all valid at least for $|x| < 1$. (The theoretical justification is provided in Section 11.8.)

11. $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$.
12. $\sum_{n=1}^{\infty} n^2x^n = \frac{x^2+x}{(1-x)^3}$.
13. $\sum_{n=1}^{\infty} n^3x^n = \frac{x^3+4x^2+x}{(1-x)^4}$.
14. $\sum_{n=1}^{\infty} n^4x^n = \frac{x^4+11x^3+11x^2+x}{(1-x)^5}$.
15. $\sum_{n=1}^{\infty} \frac{x^n}{n} = \log \frac{1}{1-x}$.
16. $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} = \frac{1}{2} \log \frac{1+x}{1-x}$.
17. $\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$.
18. $\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2!} x^n = \frac{1}{(1-x)^3}$.
19. $\sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{3!} x^n = \frac{1}{(1-x)^4}$.
20. The results of Exercises 11 through 14 suggest that there exists a general formula of the form

$$\sum_{n=1}^{\infty} n^k x^n = \frac{P_k(x)}{(1-x)^{k+1}},$$

where $P_k(x)$ is a polynomial of degree k, the term of lowest degree being x and that of highest degree being x^k. Prove this by induction, without attempting to justify the formal manipulations with the series.

21. The results of Exercises 17 through 19 suggest the more general formula

$$\sum_{n=0}^{\infty}\binom{n+k}{k}x^n = \frac{1}{(1-x)^{k+1}}, \qquad \text{where} \quad \binom{n+k}{k} = \frac{(n+1)(n+2)\cdots(n+k)}{k!}.$$

Prove this by induction, without attempting to justify the formal manipulations with the series.

22. Given that $\sum_{n=0}^{\infty} x^n/n! = e^x$ for all x, find the sums of the following series, assuming it is permissible to operate on infinite series as though they were finite sums.

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{n-1}{n!}$. (b) $\displaystyle\sum_{n=2}^{\infty}\frac{n+1}{n!}$. (c) $\displaystyle\sum_{n=2}^{\infty}\frac{(n-1)(n+1)}{n!}$.

23. (a) Given that $\sum_{n=0}^{\infty} x^n/n! = e^x$ for all x, show that

$$\sum_{n=1}^{\infty}\frac{n^2x^n}{n!} = (x^2+x)e^x,$$

assuming it is permissible to operate on these series as though they were finite sums.
(b) The sum of the series $\sum_{n=1}^{\infty} n^3/n!$ is ke, where k is a positive integer. Find the value of k. Do not attempt to justify formal manipulations.

24. Two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are called *identical* if $a_n = b_n$ for each $n \geq 1$. For example, the series

$$0 + 0 + 0 + \cdots \quad \text{and} \quad (1-1) + (1-1) + (1-1) + \cdots$$

are identical, but the series

$$1 + 1 + 1 + \cdots \quad \text{and} \quad 1 + 0 + 1 + 0 + 1 + 0 + \cdots$$

are not identical. Determine whether or not the series are identical in each of the following pairs:
(a) $1 - 1 + 1 - 1 + \cdots$ and $(2-1) - (3-2) + (4-3) - (5-4) + \cdots$.
(b) $1 - 1 + 1 - 1 + \cdots$ and $(1-1) + (1-1) + (1-1) + (1-1) + \cdots$.
(c) $1 - 1 + 1 - 1 + \cdots$ and $1 + (-1+1) + (-1+1) + (-1+1) + \cdots$.
(d) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ and $1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{8}) + \cdots$.

25. (a) Use (10.26) to prove that

$$1 + 0 + x^2 + 0 + x^4 + \cdots = \frac{1}{1-x^2} \qquad \text{if } |x| < 1.$$

Note that, according to the definition given in Exercise 24, this series is not identical to the one in (10.26) if $x \neq 0$.
(b) Apply Theorem 10.2 to the result in part (a) and to (10.25) to deduce (10.27).
(c) Show that Theorem 10.2 when applied directly to (10.25) and (10.26) does not yield (10.27). Instead, it yields the formula $\sum_{n=1}^{\infty}(x^n - x^{2n}) = x/(1-x^2)$, valid for $|x| < 1$.

EXAMPLE 1. The integral test enables us to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{converges if and only if} \quad s > 1 .$$

Taking $f(x) = x^{-s}$, we have

$$t_n = \int_1^n \frac{1}{x^s}\, dx = \begin{cases} \dfrac{n^{1-s} - 1}{1 - s} & \text{if } s \neq 1 , \\ \log n & \text{if } s = 1 . \end{cases}$$

When $s > 1$ the term $n^{1-s} \to 0$ as $n \to \infty$ and hence $\{t_n\}$ converges. By the integral test, this implies convergence of the series for $s > 1$.

When $s \le 1$, then $t_n \to \infty$ and the series diverges. The special case $s = 1$ (the *harmonic series*) was discussed earlier in Section 10.5. Its divergence was known to Leibniz.

EXAMPLE 2. The same method may be used to prove that

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^s} \quad \text{converges if and only if} \quad s > 1 .$$

(We start the sum with $n = 2$ to avoid n for which $\log n$ may be zero.)

The corresponding integral in this case is

$$t_n = \int_2^n \frac{1}{x(\log x)^s}\, dx = \begin{cases} \dfrac{(\log n)^{1-s} - (\log 2)^{1-s}}{1 - s} & \text{if } s \neq 1 , \\ \log(\log n) - \log(\log 2) & \text{if } s = 1 . \end{cases}$$

Thus $\{t_n\}$ converges if and only if $s > 1$, and hence, by the integral test, the same holds true for the series in question.

10.14 Exercises

Test the following series for convergence or divergence. In each case, give a reason for your decision.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(4n-3)(4n-1)}$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n-1}\,\log(4n+1)}{n(n+1)}$.
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2^n}$.
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.
5. $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin nx|}{n^2}$.
6. $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n}$.
7. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$.
8. $\displaystyle\sum_{n=2}^{\infty} \frac{\log n}{n\sqrt{n+1}}$.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sqrt{n}}{(n+1)^3 - 1}$.

11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^s}$.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{|a_n|}{10^n}, \quad |a_n| < 10.$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1000n + 1}$.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n \cos^2 (n\pi/3)}{2^n}$.

15. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^s}$.

16. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$.

17. $\displaystyle\sum_{n=1}^{\infty} \int_0^{1/n} \frac{\sqrt{x}}{1+x^2}\, dx$.

18. $\displaystyle\sum_{n=1}^{\infty} \int_n^{n+1} e^{-\sqrt{x}}\, dx$.

19. Assume f is a nonnegative increasing function defined for all $x \ge 1$. Use the method suggested by the proof of the integral test to show that

$$\sum_{k=1}^{n-1} f(k) \le \int_1^n f(x)\, dx \le \sum_{k=2}^{n} f(k).$$

Take $f(x) = \log x$ and deduce the inequalities

$$e\, n^n e^{-n} < n! < e\, n^{n+1} e^{-n}. \tag{10.41}$$

These give a rough estimate of the order of magnitude of $n!$. From (10.41), we may write

$$\frac{e^{1/n}}{e} < \frac{(n!)^{1/n}}{n} < \frac{e^{1/n} n^{1/n}}{e}.$$

Letting $n \to \infty$, we find that

$$\frac{(n!)^{1/n}}{n} \to \frac{1}{e} \quad \text{or} \quad (n!)^{1/n} \sim \frac{n}{e} \quad \text{as} \quad n \to \infty.$$

10.15 The root test and the ratio test for series of nonnegative terms

Using the geometric series $\sum x^n$ as a comparison series, Cauchy developed two useful tests known as the *root test* and the *ratio test.*

If $\sum a_n$ is a series whose terms (from some point on) satisfy an inequality of the form

$$0 \le a_n \le x^n, \quad \text{where} \quad 0 < x < 1, \tag{10.42}$$

a direct application of the comparison test (Theorem 10.8) tells us that $\sum a_n$ converges. The inequalities in (10.42) are equivalent to

$$0 \le a_n^{1/n} \le x; \tag{10.43}$$

hence the name *root test.*

10.16 Exercises

Test the following series for convergence or divergence and give a reason for your decision in each case.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$.
3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$.
4. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$.
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$.
6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^{2n}}$.
7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^{1/n}}$.
8. $\displaystyle\sum_{n=1}^{\infty} (n^{1/n} - 1)^n$.
9. $\displaystyle\sum_{n=1}^{\infty} e^{-n^2}$.
10. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - e^{-n^2}\right)$.
11. $\displaystyle\sum_{n=1}^{\infty} \frac{(1000)^n}{n!}$.
12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{n+1/n}}{(n + 1/n)^n}$.
13. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3[\sqrt{2} + (-1)^n]^n}{3^n}$.
14. $\displaystyle\sum_{n=1}^{\infty} r^n |\sin nx|, \qquad r > 0.$

15. Let $\{a_n\}$ and $\{b_n\}$ be two sequences with $a_n > 0$ and $b_n > 0$ for all $n \ge N$, and let $c_n = b_n - b_{n+1}a_{n+1}/a_n$. Prove that:
(a) If there is a positive constant r such that $c_n \ge r > 0$ for all $n \ge N$, then $\sum a_n$ converges.
[*Hint:* Show that $\sum_{k=N}^{n} a_k \le a_N b_N / r$.]
(b) If $c_n \le 0$ for $n \ge N$ and if $\sum 1/b_n$ diverges, then $\sum a_n$ diverges.
[*Hint:* Show that $\sum a_n$ dominates $\sum 1/b_n$.]

16. Let $\sum a_n$ be a series of positive terms. Prove *Raabe's test*: If there is an $r > 0$ and an $N \ge 1$ such that

$$\frac{a_{n+1}}{a_n} \le 1 - \frac{1}{n} - \frac{r}{n} \qquad \text{for all } n \ge N,$$

then $\sum a_n$ converges. The series $\sum a_n$ diverges if

$$\frac{a_{n+1}}{a_n} \ge 1 - \frac{1}{n} \qquad \text{for all } n \ge N.$$

[*Hint:* Use Exercise 15 with $b_{n+1} = n$.]

17. Let $\sum a_n$ be a series of positive terms. Prove *Gauss' test:* If there is an $N \ge 1$, an $s > 1$, and an $M > 0$ such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{A}{n} + \frac{f(n)}{n^s} \qquad \text{for } n \ge N,$$

where $|f(n)| \le M$ for all n, then $\sum a_n$ converges if $A > 1$ and diverges if $A \le 1$.
[*Hint:* If $A \ne 1$, use Exercise 16. If $A = 1$, use Exercise 15 with $b_{n+1} = n \log n$.]

EXAMPLES. Assume $\{b_n\}$ is any decreasing sequence of real numbers with limit 0. Taking $a_n = x^n$ in Dirichlet's test, where x is complex, $|x| = 1$, $x \neq 1$, we find that the series

$$\sum_{n=1}^{\infty} b_n x^n \tag{10.55}$$

converges. Note that Leibniz's rule for alternating series is merely the special case in which $x = -1$. If we write $x = e^{i\theta}$, where θ is real but not an integer multiple of 2π, and consider the real and imaginary parts of (10.55), we deduce that the two trigonometric series

$$\sum_{n=1}^{\infty} b_n \cos n\theta \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \sin n\theta$$

converge. In particular, when $b_n = n^{-\alpha}$, where $\alpha > 0$, we find the following series converge:

$$\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n^\alpha}, \qquad \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^\alpha}, \qquad \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^\alpha}.$$

When $\alpha > 1$, they converge absolutely since they are dominated by $\sum n^{-\alpha}$.

10.20 Exercises

In Exercises 1 through 32, determine convergence or divergence of the given series. In case of convergence, determine whether the series converges absolutely or conditionally.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$.

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n + 100}$.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$.

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right)^3$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)/2}}{2^n}$.

6. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+100}{3n+1}\right)^n$.

7. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n} + (-1)^n}$.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$.

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{1 + n^2}$.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\log (e^n + e^{-n})}$.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n \log^2 (n+1)}$.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\log (1 + 1/n)}$.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^{37}}{(n+1)!}$.

14. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \int_n^{n+1} \frac{e^{-x}}{x}\, dx$.

15. $\displaystyle\sum_{n=1}^{\infty} \sin (\log n)$.

16. $\displaystyle\sum_{n=1}^{\infty} \log \left(n \sin \frac{1}{n}\right)$.

17. $\sum_{n=1}^{\infty} (-1)^n \left(1 - n \sin \frac{1}{n}\right)$.

18. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \cos \frac{1}{n}\right)$.

19. $\sum_{n=1}^{\infty} (-1)^n \arctan \frac{1}{2n+1}$.

20. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{\pi}{2} - \arctan (\log n)\right)$.

21. $\sum_{n=1}^{\infty} \log \left(1 + \frac{1}{|\sin n|}\right)$.

22. $\sum_{n=2}^{\infty} \sin \left(n\pi + \frac{1}{\log n}\right)$.

23. $\sum_{n=1}^{\infty} \frac{1}{n(1 + 1/2 + \cdots + 1/n)}$.

24. $\sum_{n=1}^{\infty} (-1)^n \left[e - \left(1 + \frac{1}{n}\right)^n\right]$.

25. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(n + (-1)^n)^s}$.

26. $\sum_{n=1}^{\infty} (-1)^{n(n-1)/2} \left(\frac{n^{100}}{2^n}\right)$.

27. $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} 1/n & \text{if } n \text{ is a square,} \\ 1/n^2 & \text{otherwise.} \end{cases}$

28. $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} 1/n^2 & \text{if } n \text{ is odd,} \\ -1/n & \text{if } n \text{ is even.} \end{cases}$

29. $\sum_{n=1}^{\infty} \left(\sin \frac{1}{n}\right)^{3/2}$.

30. $\sum_{n=1}^{\infty} \frac{\sin (1/n)}{n}$.

31. $\sum_{n=1}^{\infty} \left(1 - n \sin \frac{1}{n}\right)$.

32. $\sum_{n=1}^{\infty} \frac{1 - n \sin (1/n)}{n}$.

In Exercises 33 through 46, describe the set of all complex z for which the series converges.

33. $\sum_{n=1}^{\infty} n^n z^n$.

34. $\sum_{n=1}^{\infty} \frac{(-1)^n z^{3n}}{n!}$.

35. $\sum_{n=0}^{\infty} \frac{z^n}{3^n}$.

36. $\sum_{n=1}^{\infty} \frac{z^n}{n^n}$.

37. $\sum_{n=1}^{\infty} \frac{(-1)^n}{z + n}$.

38. $\sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}} \log \frac{2n+1}{n}$.

39. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{5n+1}\right)^{n^2} |z|^{17n}$.

40. $\sum_{n=0}^{\infty} \frac{(z-1)^n}{(n+2)!}$.

41. $\sum_{n=1}^{\infty} \frac{(-1)^n (z-1)^n}{n}$.

42. $\sum_{n=1}^{\infty} \frac{(2z+3)^n}{n \log (n+1)}$.

43. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \left(\frac{1-z}{1+z}\right)^n$.

44. $\sum_{n=1}^{\infty} \left(\frac{z}{2z+1}\right)^n$.

45. $\sum_{n=1}^{\infty} \frac{n}{n+1} \left(\frac{z}{2z+1}\right)^n$.

46. $\sum_{n=1}^{\infty} \frac{1}{(1 + |z|^2)^n}$.

In Exercises 47 and 48, determine the set of real x for which the given series converges.

47. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n \sin^{2n} x}{n}$.

48. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n \sin^n x}{n^2}$.

In Exercises 49 through 52, the series are assumed to have real terms.

49. If $a_n > 0$ and $\sum a_n$ converges, prove that $\sum 1/a_n$ diverges.
50. If $\sum |a_n|$ converges, prove that $\sum a_n^2$ converges. Give a counterexample in which $\sum a_n^2$ converges but $\sum |a_n|$ diverges.
51. Given a convergent series $\sum a_n$, where each $a_n \geq 0$. Prove that $\sum \sqrt{a_n}\, n^{-p}$ converges if $p > \frac{1}{2}$. Give a counterexample for $p = \frac{1}{2}$.
52. Prove or disprove the following statements:
    (a) If $\sum a_n$ converges absolutely, then so does $\sum a_n^2/(1 + a_n^2)$.
    (b) If $\sum a_n$ converges absolutely, and if no $a_n = -1$, then $\sum a_n/(1 + a_n)$ converges absolutely.

*10.21 Rearrangements of series

The order of the terms in a finite sum can be rearranged without affecting the value of the sum. In 1833 Cauchy made the surprising discovery that this is not always true for infinite series. For example, consider the alternating harmonic series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + - \cdots = \log 2\,. \tag{10.56}$$

The convergence of this series to the sum log 2 was shown in Section 10.17. If we rearrange the terms of this series, taking alternately two positive terms followed by one negative term, we get a new series which can be designated as follows:

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + + - \cdots. \tag{10.57}$$

Each term which occurs in the alternating harmonic series occurs exactly once in this rearrangement, and vice versa. But we can easily prove that this new series has a sum greater than log 2. We proceed as follows:

Let t_n denote the nth partial sum of (10.57). If n is a multiple of 3, say $n = 3m$, the partial sum t_{3m} contains $2m$ positive terms and m negative terms and is given by

$$t_{3m} = \sum_{k=1}^{2m} \frac{1}{2k-1} - \sum_{k=1}^{m} \frac{1}{2k} = \left(\sum_{k=1}^{4m} \frac{1}{k} - \sum_{k=1}^{2m} \frac{1}{2k}\right) - \frac{1}{2}\sum_{k=1}^{m} \frac{1}{k} = \sum_{k=1}^{4m} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{2m} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{m} \frac{1}{k}\,.$$

In each of the last three sums, we use the asymptotic relation

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + C + o(1) \qquad \text{as} \quad n \to \infty\,,$$

to obtain

$$\begin{aligned} t_{3m} &= (\log 4m + C + o(1)) - \tfrac{1}{2}(\log 2m + C + o(1)) - \tfrac{1}{2}(\log m + C + o(1)) \\ &= \tfrac{3}{2}\log 2 + o(1)\,. \end{aligned}$$

Proof. Define a_n^+ and a_n^- as indicated in (10.59). Both series $\sum a_n^+$ and $\sum a_n^-$ diverge since $\sum a_n$ is conditionally convergent. We rearrange $\sum a_n$ as follows:

Take, in order, just enough positive terms a_n^+ so that their sum exceeds S. If p_1 positive terms are required, we have

$$\sum_{n=1}^{p_1} a_n > S \qquad \text{but} \qquad \sum_{n=1}^{q} a_n \le S \qquad \text{if} \quad q < p_1 .$$

This is always possible since the partial sums of $\sum a_n^+$ tend to $+\infty$. To this sum we add just enough negative terms a_n^-, say n_1 negative terms, so that the resulting sum is less than S. This is possible since the partial sums of a_n^- tend to $-\infty$. Thus, we have

$$\sum_{n=1}^{p_1} a_n^+ + \sum_{n=1}^{n_1} a_n^- < S \qquad \text{but} \qquad \sum_{n=1}^{p_1} a_n^+ + \sum_{n=1}^{m} a_n^- \ge S \qquad \text{if} \quad m < n_1 .$$

Now we repeat the process, adding just enough new positive terms to make the sum exceed S, and then just enough new negative terms to make the sum less than S. Continuing in this way, we obtain a rearrangement $\sum b_n$. Each partial sum of $\sum b_n$ differs from S by at most one term a_n^+ or a_n^-. But $a_n \to 0$ as $n \to \infty$ since $\sum a_n$ converges, so the partial sums of $\sum b_n$ tend to S. This proves that the rearranged series $\sum b_n$ converges and has sum S, as asserted.

10.22 Miscellaneous review exercises

1. (a) Let $a_n = \sqrt{n+1} - \sqrt{n}$. Compute $\lim_{n\to\infty} a_n$.
   (b) Let $a_n = (n+1)^c - n^c$, where c is real. Determine those c for which the sequence $\{a_n\}$ converges and those for which it diverges. In case of convergence, compute the limit of the sequence. Remember that c can be positive, negative, or zero.
2. (a) If $0 < x < 1$, prove that $(1 + x^n)^{1/n}$ approaches a limit as $n \to \infty$ and compute this limit.
   (b) Given $a > 0$, $b > 0$, compute $\lim_{n\to\infty}(a^n + b^n)^{1/n}$.
3. A sequence $\{a_n\}$ is defined recursively in terms of a_1 and a_2 by the formula

$$a_{n+1} = \frac{a_n + a_{n-1}}{2} \qquad \text{for} \quad n \ge 2 .$$

   (a) Assuming that $\{a_n\}$ converges, compute the limit of the sequence in terms of a_1 and a_2. The result is a weighted arithmetic mean of a_1 and a_2.
   (b) Prove that for every choice of a_1 and a_2 the sequence $\{a_n\}$ converges. You may assume that $a_1 < a_2$. [*Hint:* Consider $\{a_{2n}\}$ and $\{a_{2n+1}\}$ separately.]
4. A sequence $\{x_n\}$ is defined by the following recursion formula:

$$x_1 = 1 , \qquad x_{n+1} = \sqrt{1 + x_n} .$$

   Prove that the sequence converges and find its limit.
5. A sequence $\{x_n\}$ is defined by the following recursion formula:

$$x_0 = 1 , \qquad x_1 = 1 , \qquad \frac{1}{x_{n+2}} = \frac{1}{x_{n+1}} + \frac{1}{x_n} .$$

   Prove that the sequence converges and find its limit.

6. Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that for each n we have

$$e^{a_n} = a_n + e^{b_n}$$

(a) Show that $a_n > 0$ implies $b_n > 0$.
(b) If $a_n > 0$ for all n and if $\sum a_n$ converges, show that $\sum (b_n/a_n)$ converges.

In Exercises 7 through 11, test the given series for convergence.

7. $\sum_{n=1}^{\infty} (\sqrt{1 + n^2} - n)$.

8. $\sum_{n=1}^{\infty} n^s(\sqrt{n+1} - 2\sqrt{n} + \sqrt{n-1})$.

9. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$.

10. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$.

11. $\sum_{n=1}^{\infty} a_n$, where $a_n = 1/n$ if n is odd, $a_n = 1/n^2$ if n is even.

12. Show that the infinite series

$$\sum_{n=0}^{\infty} (\sqrt{n^a + 1} - \sqrt{n^a})$$

converges for $a > 2$ and diverges for $a = 2$.

13. Given $a_n > 0$ for each n. For each of the following statements, give a proof or exhibit a counterexample.
(a) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} a_n^2$ diverges.
(b) If $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} a_n/n$ converges.

14. Find all real c for which the series $\sum_{n=1}^{\infty} (n!)^c/(3n)!$ converges.

15. Find all integers $a \geq 1$ for which the series $\sum_{n=1}^{\infty} (n!)^3/(an)!$ converges.

16. Let $n_1 < n_2 < n_3 < \cdots$ denote those positive integers that do not involve the digit 0 in their decimal representations. Thus $n_1 = 1, n_2 = 2, \ldots, n_9 = 9, n_{10} = 11, \ldots, n_{18} = 19, n_{19} = 21$, etc. Show that the series of reciprocals $\sum_{k=1}^{\infty} 1/n_k$ converges and has a sum less than 90.

[*Hint:* Dominate the series by $9 \sum_{n=0}^{\infty} (9/10)^n$.]

17. If a is an arbitrary real number, let $s_n(a) = 1^a + 2^a + \cdots + n^a$. Determine the following limit:

$$\lim_{n\to\infty} \frac{s_n(a+1)}{n s_n(a)}.$$

(Consider both positive and negative a, as well as $a = 0$.)

18. (a) If p and q are fixed integers, $p \geq q \geq 1$, show that

$$\lim_{n\to\infty} \sum_{k=qn}^{pn} \frac{1}{k} = \log \frac{p}{q}.$$

(b) The following series is a rearrangement of the alternating harmonic series in which there appear, alternately, three positive terms followed by two negative terms:

$$1 + \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} - \tfrac{1}{8} + + + - - \cdots.$$

Show that the series converges and has sum $\log 2 + \frac{1}{2} \log \frac{3}{2}$.

[*Hint:* Consider the partial sum s_{5n} and use part (a).]

But $\int_1^\infty e^{-1/u}u^{-s-1}\,du$ converges for $s > 0$ by comparison with $\int_1^\infty u^{-s-1}\,du$. Therefore the integral $\int_{0+}^1 e^{-t}t^{s-1}\,dt$ converges for $s > 0$. When $s > 0$, the sum in (10.65) is denoted by $\Gamma(s)$. The function Γ so defined is called the *gamma function*, first introduced by Euler in 1729. It has the interesting property that $\Gamma(n + 1) = n!$ when n is any integer ≥ 0. (See Exercise 19 of Section 10.24 for an outline of the proof.)

The convergence tests given in Theorems 10.23 through 10.25 have straightforward analogs for improper integrals of the second kind. The reader should have no difficulty in formulating these tests for himself.

10.24 Exercises

In each of Exercises 1 through 10, test the improper integral for convergence.

1. $\displaystyle\int_0^\infty \frac{x}{\sqrt{x^4+1}}\,dx.$

2. $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx.$

3. $\displaystyle\int_0^\infty \frac{1}{\sqrt{x^3+1}}\,dx.$

4. $\displaystyle\int_0^\infty \frac{1}{\sqrt{e^x}}\,dx.$

5. $\displaystyle\int_{0+}^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx.$

6. $\displaystyle\int_{0+}^1 \frac{\log x}{\sqrt{x}}\,dx.$

7. $\displaystyle\int_{0+}^{1-} \frac{\log x}{1-x}\,dx.$

8. $\displaystyle\int_{-\infty}^\infty \frac{x}{\cosh x}\,dx.$

9. $\displaystyle\int_{0+}^{1-} \frac{dx}{\sqrt{x}\log x}.$

10. $\displaystyle\int_2^\infty \frac{dx}{x(\log x)^s}.$

11. For a certain real C the integral

$$\int_2^\infty \left(\frac{Cx}{x^2+1} - \frac{1}{2x+1}\right) dx$$

converges. Determine C and evaluate the integral.

12. For a certain real C, the integral

$$\int_1^\infty \left(\frac{x}{2x^2+2C} - \frac{C}{x+1}\right) dx$$

converges. Determine C and evaluate the integral.

13. For a certain real C, the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{1+2x^2}} - \frac{C}{x+1}\right) dx$$

converges. Determine C and evaluate the integral.

14. Find the values of a and b such that

$$\int_1^\infty \left(\frac{2x^2+bx+a}{x(2x+a)} - 1\right) dx = 1 .$$

15. For what values of the constants a and b will the following limit exist and be equal to 1?

$$\lim_{p\to+\infty} \int_{-p}^{p} \frac{x^3 + ax^2 + bx}{x^2 + x + 1}\, dx .$$

16. (a) Prove that

$$\lim_{h\to 0+} \left(\int_{-1}^{-h} \frac{dx}{x} + \int_{h}^{1} \frac{dx}{x} \right) = 0 \quad \text{and that} \quad \lim_{h\to+\infty} \int_{-h}^{h} \sin x\, dx = 0 .$$

(b) Do the following improper integrals converge or diverge?

$$\int_{-1}^{1} \frac{dx}{x} ; \qquad \int_{-\infty}^{\infty} \sin x\, dx .$$

17. (a) Prove that the integral $\int_{0+}^{1} (\sin x)/x\, dx$ converges.
(b) Prove that $\lim_{x\to 0+} x\int_x^1 (\cos t)/t^2\, dt = 1$.
(c) Does the integral $\int_{0+}^{1} (\cos t)/t^2\, dt$ converge or diverge?

18. (a) If f is monotonic decreasing for all $x \geq 1$ and if $f(x) \to 0$ as $x \to +\infty$, prove that the integral $\int_1^\infty f(x)\, dx$ and the series $\sum f(n)$ both converge or both diverge.

[*Hint:* Recall the proof of the integral test.]

(b) Give an example of a nonmonotonic f for which the series $\sum f(n)$ converges and the integral $\int_1^\infty f(x)\, dx$ diverges.

19. Let $\Gamma(s) = \int_{0+}^{\infty} t^{s-1}e^{-t}\, dt$, if $s > 0$. (The gamma function.) Use integration by parts to show $\Gamma(s + 1) = s\Gamma(s)$. Then use induction to prove that $\Gamma(n + 1) = n!$ if n is a positive integer.

Each of Exercises 20 through 25 contains a statement, not necessarily true, about a function f defined for all $x \geq 1$. In each of these exercises, n denotes a positive integer, and I_n denotes the integral $\int_1^n f(x)\, dx$, which is always assumed to exist. For each statement either give a proof or provide a counterexample.

20. If f is monotonic decreasing and if $\lim_{n\to\infty} I_n$ exists, then the integral $\int_1^\infty f(x)\, dx$ converges.
21. If $\lim_{x\to\infty} f(x) = 0$ and $\lim_{n\to\infty} I_n = A$, then $\int_1^\infty f(x)\, dx$ converges and has the value A.
22. If the sequence $\{I_n\}$ converges, then the integral $\int_1^\infty f(x)\, dx$ converges.
23. If f is positive and if $\lim_{n\to\infty} I_n = A$, then $\int_1^\infty f(x)\, dx$ converges and has the value A.
24. Assume $f'(x)$ exists for each $x \geq 1$ and suppose there is a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \geq 1$. If $\lim_{n\to\infty} I_n = A$, then the integral $\int_1^\infty f(x)\, dx$ converges and has the value A.
25. If $\int_1^\infty f(x)\, dx$ converges, then $\lim_{x\to\infty} f(x) = 0$.