

ON THE TRACE OF A WISHART

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ABSTRACT

A derivation based on spectral decomposition allows specifying the characteristic function of the trace of a singular or nonsingular, central or noncentral, true or pseudo-Wishart. The trace equals a weighted sum of noncentral chi-squared random variables and constants. We describe computational methods.

1. INTRODUCTION

Johnson and Kotz (1972) gave a thorough review of many results about the Wishart. Arnold (1984, pp. 314-317) defined the noncentral Wishart in terms of the matrix normal distribution. Mathai and Provost (1992, p.258) provided definitions for singular and nonsingular, true and pseudo-Wisharts.

Saw (1973) and Shah and Khatri (1974) provided proofs for de Waal's (1972) conjecture about the expected values of elementary symmetric functions of a noncentral nonsingular true Wishart matrix. Mathai (1980) gave the moments,

cumulants and asymptotic distribution of the trace of a noncentral nonsingular true Wishart matrix. Mathai and Pillai (1982) derived the exact density of the trace both in terms of confluent hypergeometric functions and in terms of zonal polynomials. They noted in passing that the trace could also be represented as a weighted sum of noncentral chi-squared random variables, but did not give an explicit form. Kourouklis and Moschopoulos (1985) gave the distribution of the trace of a noncentral nonsingular Wishart using a single gamma series representation, by inverting the moment generating function. Muller and Barton (1989, Appendix 1; 1991) provided the explicit form, as well as a proof. Their result holds for *nonsingular* true and pseudo-Wisharts. Wong and Liu (1995) suggested calculating the moments of the trace of any Wishart using tensor differential forms.

We extend the result of Muller and Barton (1989; 1991) to include *singular* Wisharts. Note that the density of a singular or pseudo-Wishart does not exist. The traces of singular true and pseudo-Wisharts are important in the study of power in general linear multivariate models with both fixed and random predictors.

2. RESULTS

With m_j an $r \times 1$ vector, and $M = [m_1 \ m_2 \ \cdots \ m_N]$, define $\text{vec}(M) = [m'_1 \ m'_2 \ \cdots \ m'_N]'$ (Graybill, 1983, p. 309). Define $Dg(m_j)$ as the $r \times r$ matrix with i, i element $m_{j,i}$, and zero elsewhere (Stewart, 1973, p.22).

Definition 1. Let N and p be positive integers. Consider the constant matrices M ($N \times p$), Σ ($p \times p$, symmetric, positive semidefinite), Ξ ($N \times N$, symmetric, positive semidefinite), and the random matrix Y ($N \times p$). Then $\text{vec}(Y) = N_{Np}[\text{vec}(M), \Sigma \otimes \Xi]$ if and only if Y has a matrix normal distribution, written $Y = N_{N,p}(M, \Xi, \Sigma)$ (Arnold, 1981, p. 311). Here $\Sigma \otimes \Xi = \{\sigma_{ij} \cdot \Xi\}$. If Σ or Ξ are less than full rank, Y has a singular matrix normal distribution, written $Y = SN_{N,p}(M, \Xi, \Sigma)$. Write $Y = (S)N_{N,p}(M, \Xi, \Sigma)$ when Y has a matrix normal distribution which may or may not be singular.

Definition 2. Let M be an $N \times p$ constant matrix, $\Delta = M'M$, Σ a symmetric, positive semidefinite, constant $p \times p$ matrix, and $Z = (S)N_{N,p}(M, I_N, \Sigma)$. Then $W = Z'Z$ is a Wishart matrix, indicated

$$W = W_p(N, \Sigma, \Delta).$$

W is said to have a Wishart distribution if $0 < p \leq N$ and Σ positive definite and hence of full rank, a pseudo-Wishart distribution if $0 < N < p$ and Σ positive definite, a singular Wishart distribution if $0 < p \leq N$ and Σ is less than full rank, and a singular pseudo-Wishart distribution if $0 < N < p$ and Σ is less than full rank. These are central Wisharts if $M = 0$, and noncentral otherwise.

Lemma. Let Σ be a positive semidefinite symmetric matrix and let $\Delta = M' M$. If $W = W_p(N, \Sigma, \Delta)$, then there exists a matrix $Y = (S)N_{N,p}(M, I_N, \Sigma)$ such that $Y' Y = W$.

Proof. Let $Z = N_{N,p}(0, I_N, I_p)$. Use the spectral decomposition to write $\Sigma = V Dg(\lambda) V'$, and define $F = V Dg(\lambda)^{\frac{1}{2}}$. Let $Y = Z F' + M$. Then $Y = (S)N_{N,p}(M, I, \Sigma)$ (Arnold, 1981, p.312, Theorem 17.2d) and hence $Y' Y$ has the desired distribution.

Theorem. Let ν, p and p_* be non-negative. Let A be a constant $N \times N$ idempotent matrix of rank $\nu \geq p$. Let Σ be a constant, symmetric, positive semidefinite matrix of rank $p_* \leq p$, with spectral decomposition $\Sigma = V Dg(\lambda) V'$, with $V' V = I_p$. Let λ_+ be the $p_* \times 1$ vector of strictly positive eigenvalues. Without loss of generality, write

$$Dg(\lambda) = \begin{bmatrix} Dg(\lambda_+) & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} p_* \times (p-p_*) \\ (p-p_*) \times (p-p_*) \end{matrix}$$

Let δ_k be a $p \times 1$ vector with a 1 in the k^{th} row and 0 elsewhere. Let $Y = (S)N_{N,p}(M, I_N, \Sigma)$, and $M_{*k} = M V \delta_k$. Then

$$\text{tr}(Y' A Y) = \sum_{k=1}^{p_*} \lambda_k \chi^2(\nu, M'_{*k} A M_{*k}) + \sum_{k=p_*+1}^p M'_{*k} A M_{*k}$$

Proof. $Y V = (S)N_{N,p}(M V, I_N, Dg(\lambda))$, and thus $Y V \delta_k$ is independent of $Y V \delta_{k'}$ for $k \neq k'$, since $Dg(\lambda) \otimes I_N$ is diagonal. For each $k \leq p_*$, $Y V \delta_k = N_{N,1}(M_{*k}, I_N, \lambda_k)$, and thus $\delta'_k V' Y' A Y V \delta_k = \lambda_k \chi^2(\nu, M'_{*k} A M_{*k})$. For each k such that $p_* < k \leq p$, $Y V \delta_k = (S)N_{N,1}(M_{*k}, I_N, 0)$. This implies that $\text{Pr}\{Y V \delta_k = M_{*k}\} = 1$, and that $\text{Pr}\{\delta'_k V' Y' A Y V \delta_k = M'_{*k} A M_{*k}\} = 1$. The result then follows since

$$\text{tr}(Y' A Y) = \text{tr}(V' Y' A Y V) = \sum_{k=1}^p \delta'_k V' Y' A Y V \delta_k$$

Corollary 1. The characteristic function of the trace of a Wishart may be written as

$$\phi(t) = \exp \left[\sum_{k=1}^{p_*} \frac{\lambda_k M'_{*k} A M_{*k} i t}{(1 - 2\lambda_k i t)} + \sum_{k=p_*+1}^p i t M'_{*k} A M_{*k} \right] \prod_{k=1}^{p_*} (1 - 2\lambda_k i t)^{-\nu/2}.$$

Corollary 2. The mean and the variance of the trace of a Wishart are

$$\mathcal{E}[\text{tr}(Y'AY)] = \nu \sum_{k=1}^{p_*} \lambda_k + \sum_{k=1}^{p_*} \lambda_k M'_{*k} A M_{*k} + \sum_{k=p_*+1}^p M'_{*k} A M_{*k}$$

and

$$V[\text{tr}(Y'AY)] = \sum_{k=1}^{p_*} \lambda_k^2 (2\nu + 4M'_{*k} A M_{*k}).$$

3. COMPUTATIONAL METHODS

Observe that $\lambda_k > 0$ and

$$\Pr\{\text{tr}(Y'AY) \leq t\} = \Pr\left\{ \sum_{k=1}^{p_*} \lambda_k \chi^2(\nu, M'_{*k} A M_{*k}) \leq t - \sum_{k=p_*+1}^p M'_{*k} A M_{*k} \right\}.$$

One can use this characterization to compute the cumulative distribution function of the trace of a Wishart. Two different methods apply. The first is an approximation, and the second a numerical integration which allows specifying the degree of accuracy. Muller and Barton (1989; 1991) described how to approximate a positively weighted sum of noncentral χ^2 variables with a single scaled noncentral χ^2 . Davies (1980) gave a computer algorithm that employs the inverse Mellin transform to invert the characteristic function of a sum of noncentral χ^2 variables.

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