Populations and Samples
Bios 662

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Random Variables

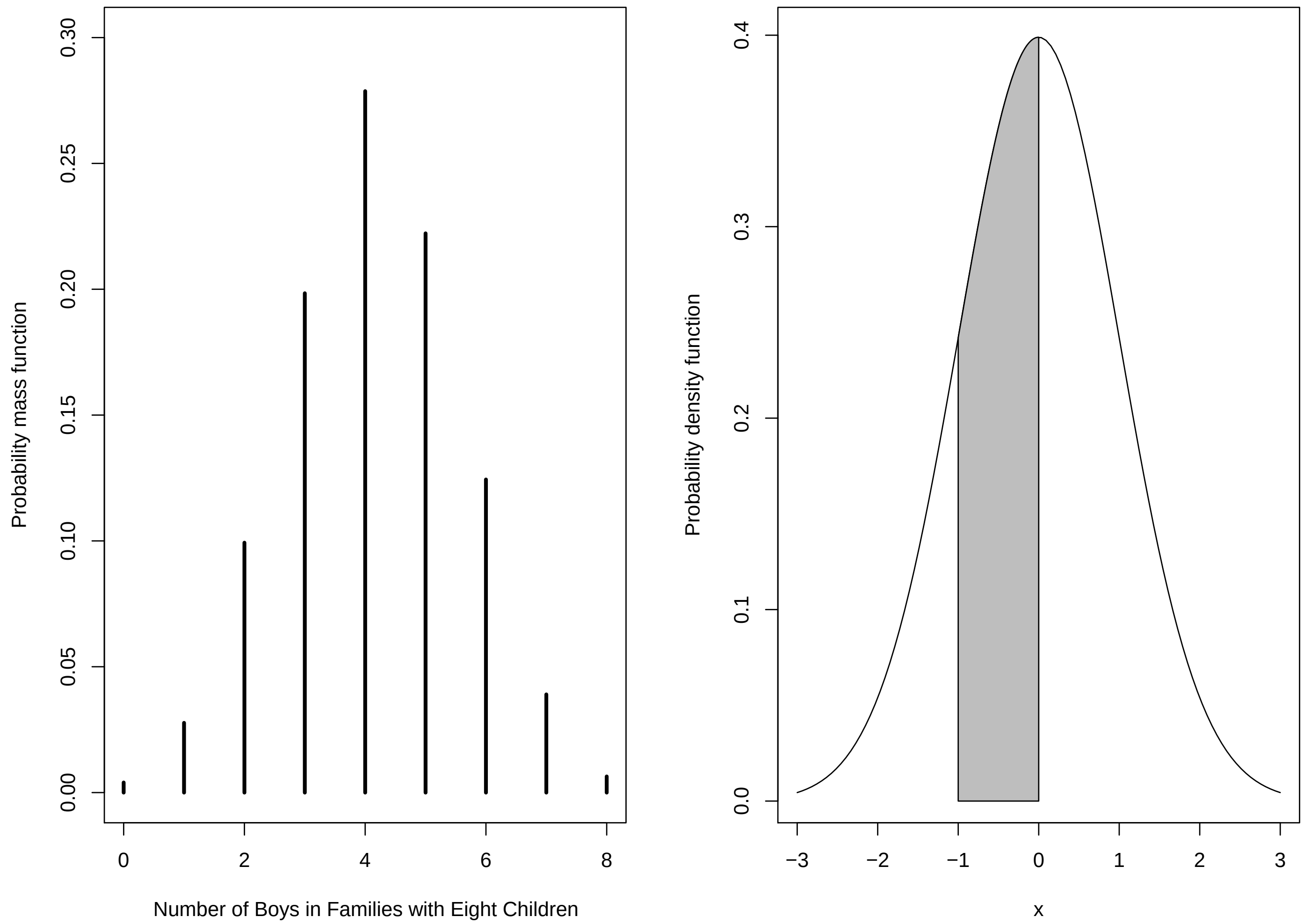
- *Random sample*: result of independently selecting elements at random from a population
- Def 4.8 A *random variable* is a variable associated with a random sample

  PV Rao (p 786, 1998) A *rv* is a variable whose value is determined by the observed characteristics of an item randomly selected from a popluation

Probability Functions

- Def 4.9 The *probability mass function* is a function that for each possible value of a discrete rv takes on the probability of that value occurring
- Def 4.10 The *probability density function* is a curve that specifices, by means of the area under the curve over an interval, the probability that a contiunous rv falls within the interval

Probability Functions



Cumulative Distribution Function

- Def 4.9 The *cumulative distribution function* for a rv X is
$$F(x) = \Pr[X \le x]$$
- If X is discrete,
$$F(x) = \sum_{y \le x} p_X(y)$$
where p_X is the pmf of X
- If X is continuous,
$$F(x) = \int_{-\infty}^{x} f(y)dy$$
where f is the pdf of X

Population quantile

- Intuitive defintion:

  The p^{th} quantile of X, say ζ_p, should be such that

  $$F(\zeta_p) = \Pr[X \leq \zeta_p] = p$$

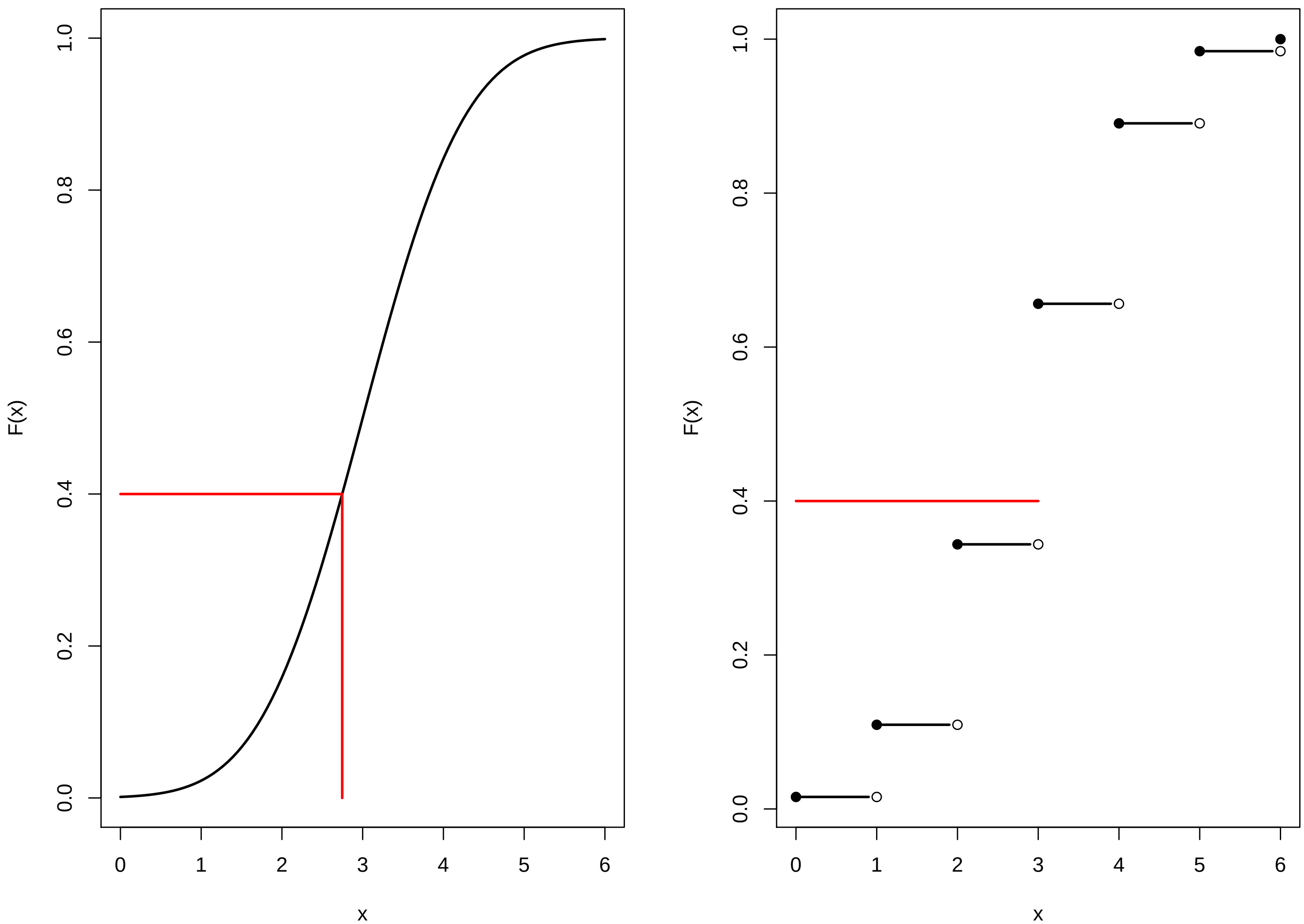
- Formally:

  $$\zeta_p = \inf\{x : F(x) \geq p\}$$

- If F is continuous

  $$F(\zeta_p) = p$$

Quantiles: Example



Mean and Variance

- Mean or expected value of X
  - If X is discrete,

$$\mu = E(X) = \sum_y y p_X(y)$$

  - If X is continuous,

$$\mu = E(X) = \int_{-\infty}^{\infty} y f(y) dy$$

- Variance

$$\sigma^2 = Var(X) = E\{(X - \mu)^2\}$$

Skewness and Kurtosis

- Skewness

$$\alpha_3 = \frac{E\{(X-\mu)^3\}}{\sigma^3}$$

- Kurtosis

$$\alpha_4 = \frac{E\{(X-\mu)^4\}}{\sigma^4}$$

Parameters and Statistics

- Definition: A *parameter* is a numerical characteristic of a population
- Definition: A *statistic* is a numerical characteristic of a sample
- Notation: Greek letters typically denote parameters; English letters denote statistics
- Example:

  $\mu =$ population mean; σ^2 population variance

  $\bar{Y}$ sample mean; s^2 sample variance

Parameters and Statistics

- Parameters are fixed constants
- Statistics are random variables
- Statistics have probability distributions
- We will use statistics and probability theory to draw conclusions (inference) about parameters

Sampling Distributions

- Defintion 4.15 The probability function of a statistic is called the *sampling distribution of the statistic.*

- Eg, when sampling from a population, the sample mean $\bar{Y}$ is a rv becuase its value depends on chance, namely, on which sample is obtained.

  The probability distribution of the random variable $\bar{Y}$ is called the *sampling distribution of the mean.*

Sampling Distributions

- Result 4.1 If a rv Y has a population mean μ and a population variance σ^2, the sampling distribution of the mean ($\bar{Y}$) has mean μ and variance σ^2/n
- Definition 4.16 The standard deviation of the sampling distribution is called the *standard error*
- Eg the standard error of $\bar{Y}$ is $\sigma/\sqrt{n}$

Normal or Gaussian Distribution

- PDF:

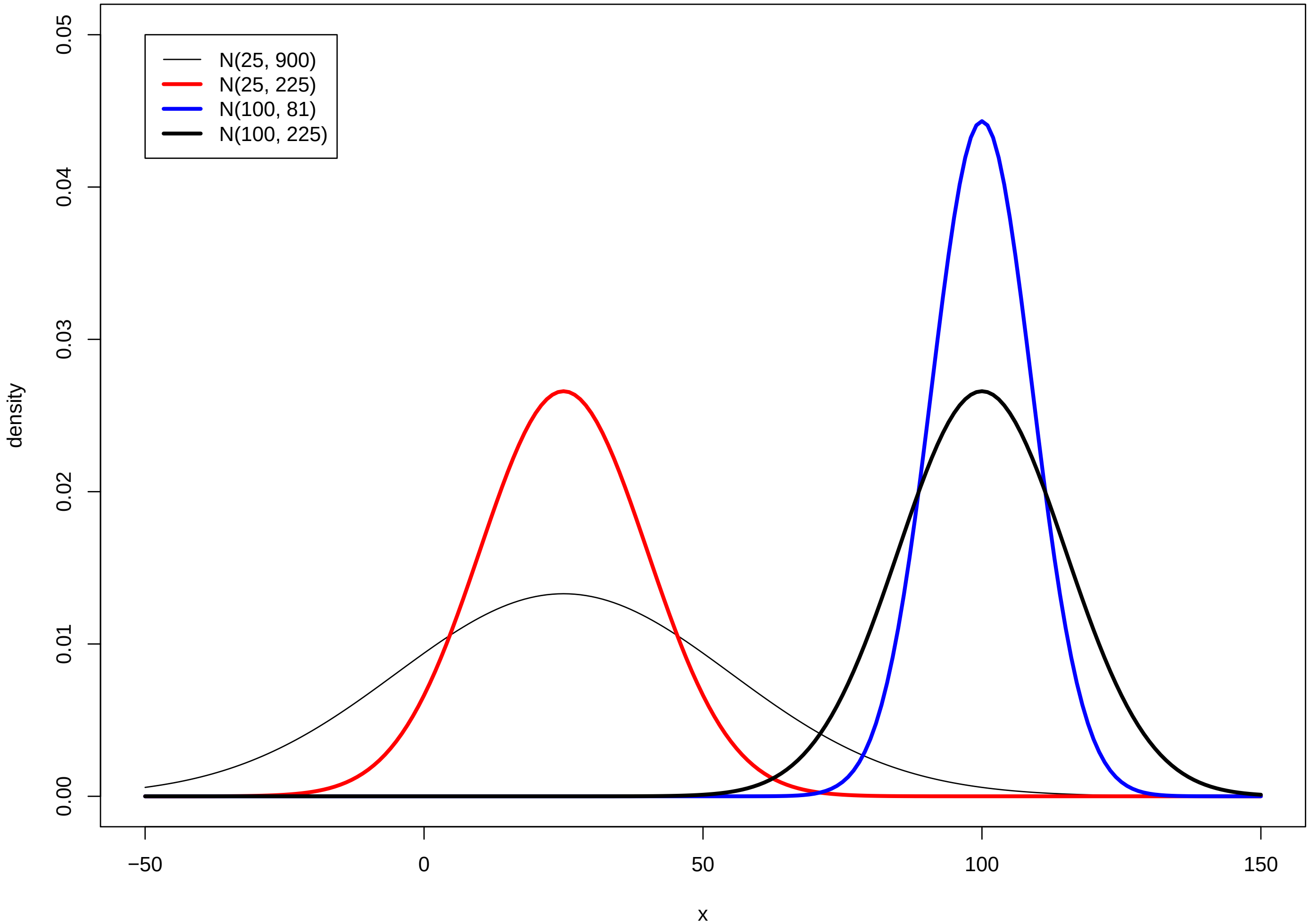
$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

- CDF:

$$F(x;\mu,\sigma) = \int_{-\infty}^{x} f(y;\mu,\sigma)dy$$

- μ mean, σ^2 variance
- $X \sim N(\mu,\sigma^2)$ [beware $X \sim N(\mu,\sigma)$]

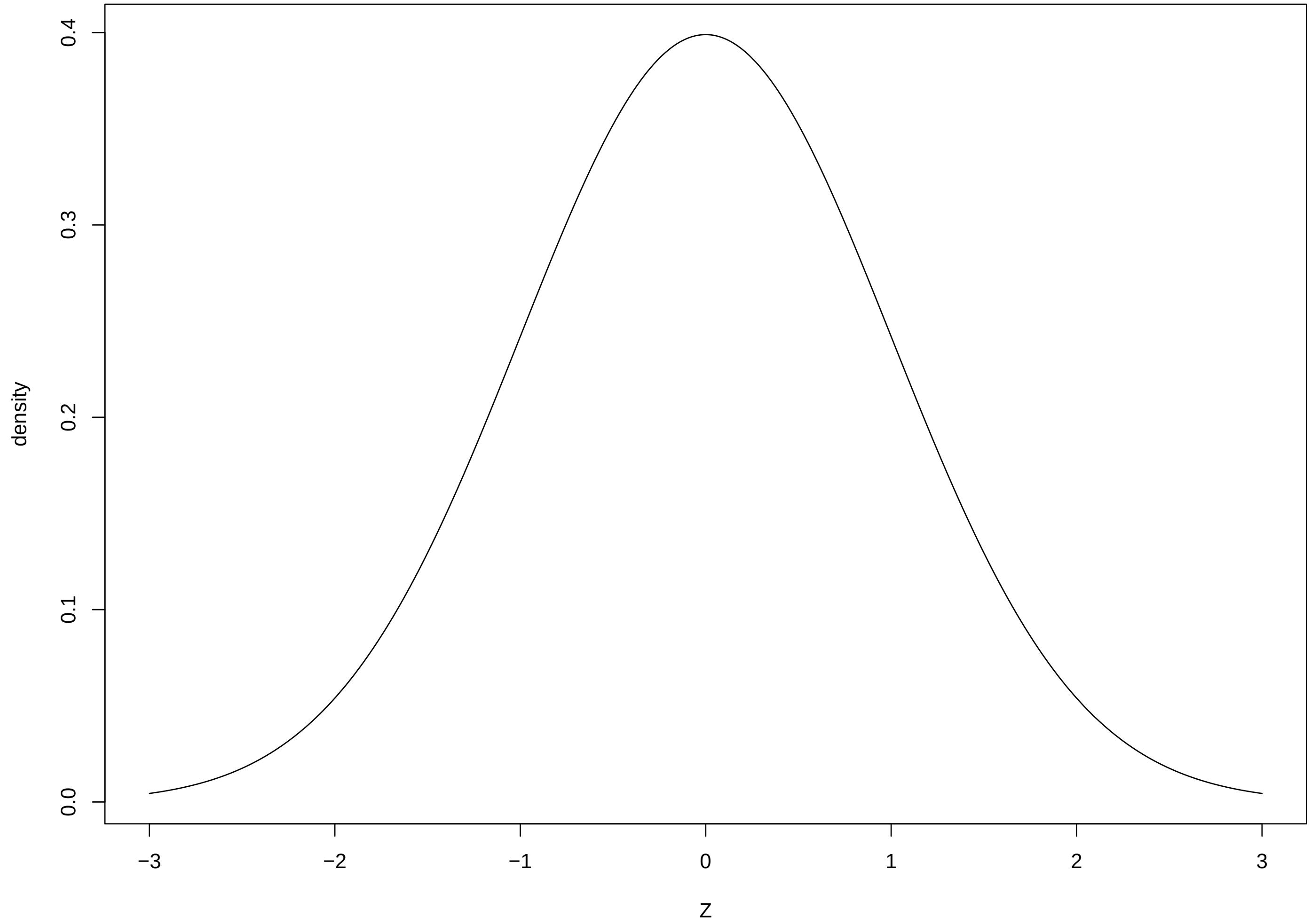
Normal Distribution



Standard Normal Distribution

- $Z \sim N(0,1)$
- PDF:
$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}z^2\}$$
- CDF:
$$\Phi(z) = \int_{-\infty}^{z} \phi(y)dy$$
- $N(0,1)$ is a *standard normal distribution*

Standard Normal Distribution



Properties of Standard Normal Distribution

- A rv w/ pdf f is *symmetric* about μ if

$$f(\mu + x) = f(\mu - x) \text{ for all } x$$

- $Z \sim N(0, 1)$ is symmetric about 0

$$\phi(z) = \phi(-z) \text{ for all } -\infty < z < \infty$$

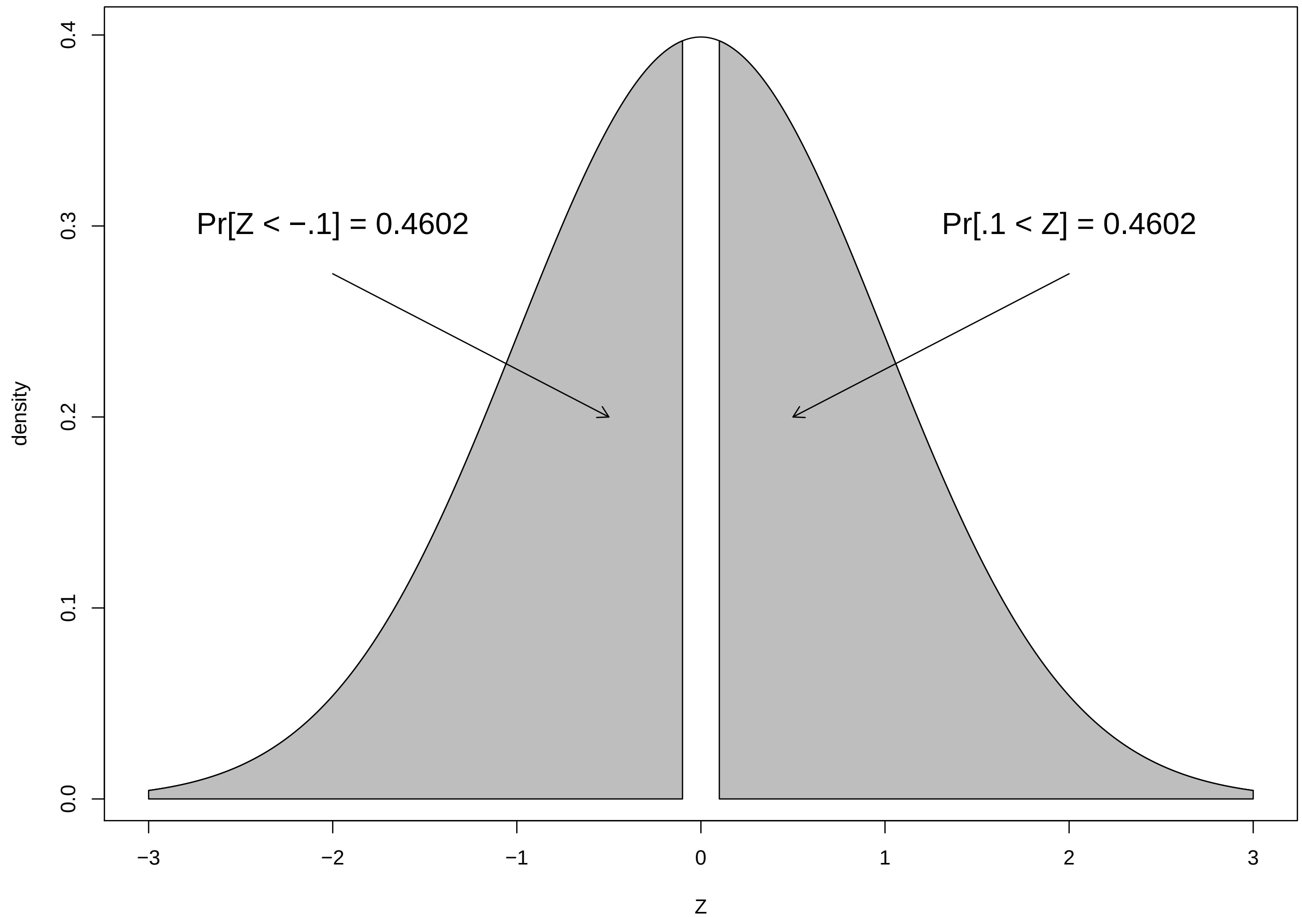
- Thus

$$\Pr[Z \leq -z] = \Pr[Z \geq z]$$

i.e.

$$\Phi(-z) = 1 - \Phi(z)$$

Standard Normal Distribution



Standard Normal Distribution

- R

```
> pnorm(-.1,0,1)
[1] 0.4601722
> 1-pnorm(.1,0,1)
[1] 0.4601722
>  qnorm(0.4601722,0,1)
[1] -0.0999999
```

Standard Normal Distribution

- SAS

```
data;
  x=probnorm(-.1);
  y=cdf('NORMAL',-.1,0,1);
  z=quantile('NORMAL',0.4601722);
  run;

proc print; run;

Obs        x           y            z
 1      0.46017     0.46017     -0.100000
```

Properties of a Random Variable

- Let X be a random variable
- Suppose $Y = aX + b$ where a and b are constants
- Then

$$E(Y) = aE(X) + b$$

$$Var(Y) = a^2 Var(X)$$

- If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, then

$$Y \sim N(a\mu + b, (a\sigma)^2)$$

Conversion to Standard Normal

- Suppose $Y \sim N(\mu, \sigma^2)$
- Let
$$Z = \frac{Y - \mu}{\sigma}$$
- Then
$$Z \sim N(0, 1)$$
- In words: a normal random variable can be standardized by subtracting its mean and dividing by its standard deviation

Computation of Probabilities

- Suppose $Y \sim N(\mu, \sigma^2)$
- Let
$$Z = \frac{Y - \mu}{\sigma}$$
- Then
$$\begin{aligned}\Pr[a < Y < b] &= \Pr[\tfrac{a-\mu}{\sigma} < Z < \tfrac{b-\mu}{\sigma}] \\ &= \Phi(\tfrac{b-\mu}{\sigma}) - \Phi(\tfrac{a-\mu}{\sigma})\end{aligned}$$

Table 1 (p 818 text): Standard normal distribution.

Let Z be a normal random variable with mean zero and variance one. For selected values of z, three values are tabled: (1) the two-sided p-value, or $\Pr[|Z| \geq z]$; (2) the one-sided p-value, or $\Pr[Z \geq z]$; and (3) the cumulative distribution function at z, or $\Pr[Z \leq z]$.

z	Two sided	One sided	Cumu. dist.
0.00	1.0000	.5000	.5000
0.05	.9601	.4801	.5199
0.10	.9203	.4602	.5398
0.15	.8808	.4404	.5596
0.20	.8415	.4207	.5793
0.25	.8026	.4013	.5987
0.30	.7642	.3821	.6179
0.35	.7263	.3632	.6368
0.40	.6892	.3446	.6554
0.45	.6527	.3264	.6736
0.50	.6171	.3085	.6915
$\vdots$			
1.00	.3173	.1587	.8413
1.33	.1835	.0918	.9082
1.64	.1010	.0505	.9495
1.96	.0500	.0250	.9750
2.00	.0455	.0288	.9772
2.58	.0099	.0049	.9951

Example

- Intraocular pressure (IP) is used to diagnose glaucoma
- Assume IP is normally distributed with mean $\mu = 16$ mmHg and variance $\sigma^2 = 9$ mmHg
- If pressure greater than 20 mmHg is considered abnormal, what proportion of the population is abnormal?

$$
\begin{aligned}
\Pr[X > 20] &= \Pr[\tfrac{X-16}{3} > \tfrac{20-16}{3}] \\
&= \Pr[Z > 1.33] = 1 - \Phi(1.33) \\
&= 1 - 0.9082 = 0.0918
\end{aligned}
$$

Example (continued)

- What proportion of the population has IP between 4 and 18?

$$\begin{aligned}
\Pr[4 < X < 18] &= \Pr[\tfrac{4-16}{3} < \tfrac{X-16}{3} < \tfrac{18-16}{3}] \\
&= \Pr[-4 < Z < 2/3] \\
&= \Phi(2/3) - \Phi(-4) \\
&= \Phi(2/3) - 1 + \Phi(4) \\
&= 0.7486 - 1 + 0.9999 = 0.7485
\end{aligned}$$

Assessing Normality

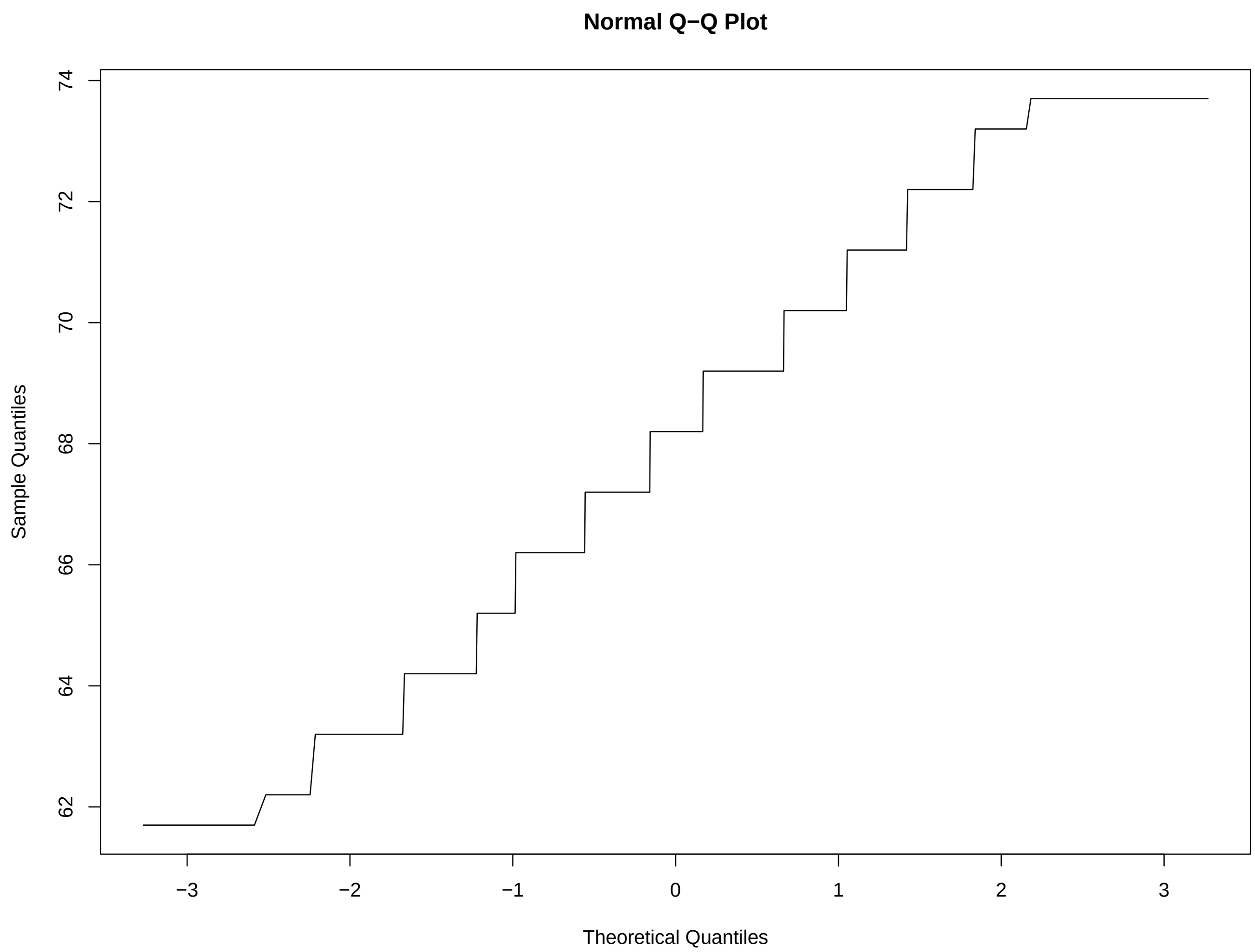
- How do we assess whether the normal distribution model fits a particular set of data?
- One graphical approach: quantile-quantile (QQ) plot
- Plot quantiles of the observed data distribution versus the quantiles of the normal distribution
- Straight line indicates normality assumption reasonable

QQ Plot Example

- Table 4.3 from text (p. 81)

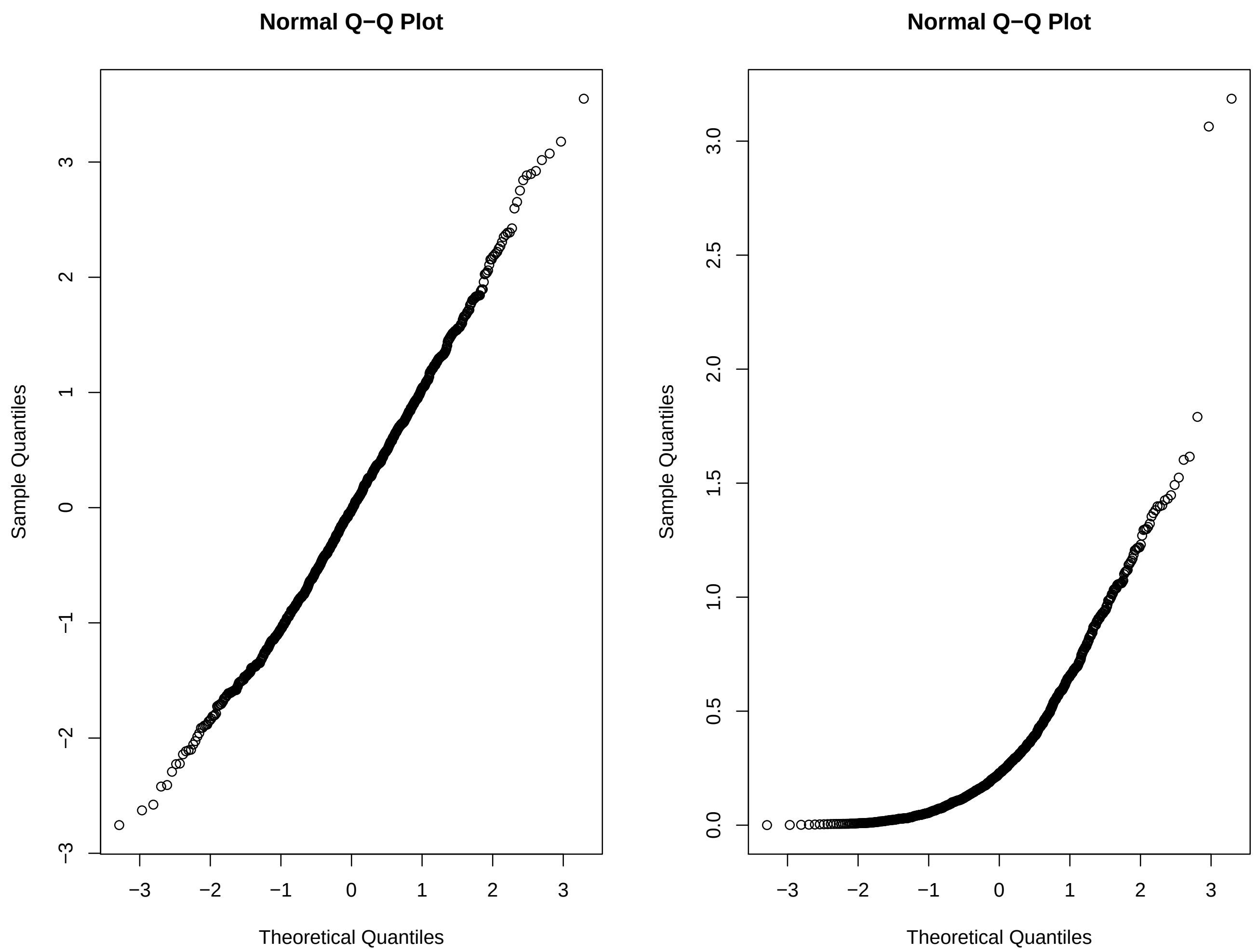
Endpoint	Freq	Cumul Pct
61.7	5	0.5
62.2	7	1.3
63.2	32	4.7
64.2	59	11.1
65.2	48	16.3
66.2	117	28.9
67.2	138	43.8
68.2	120	56.7
69.2	167	74.7
70.2	99	85.3
71.2	64	92.2
72.2	41	96.7
73.2	17	98.5
73.7	14	100

QQ Plot Example: Table 4.3 from text



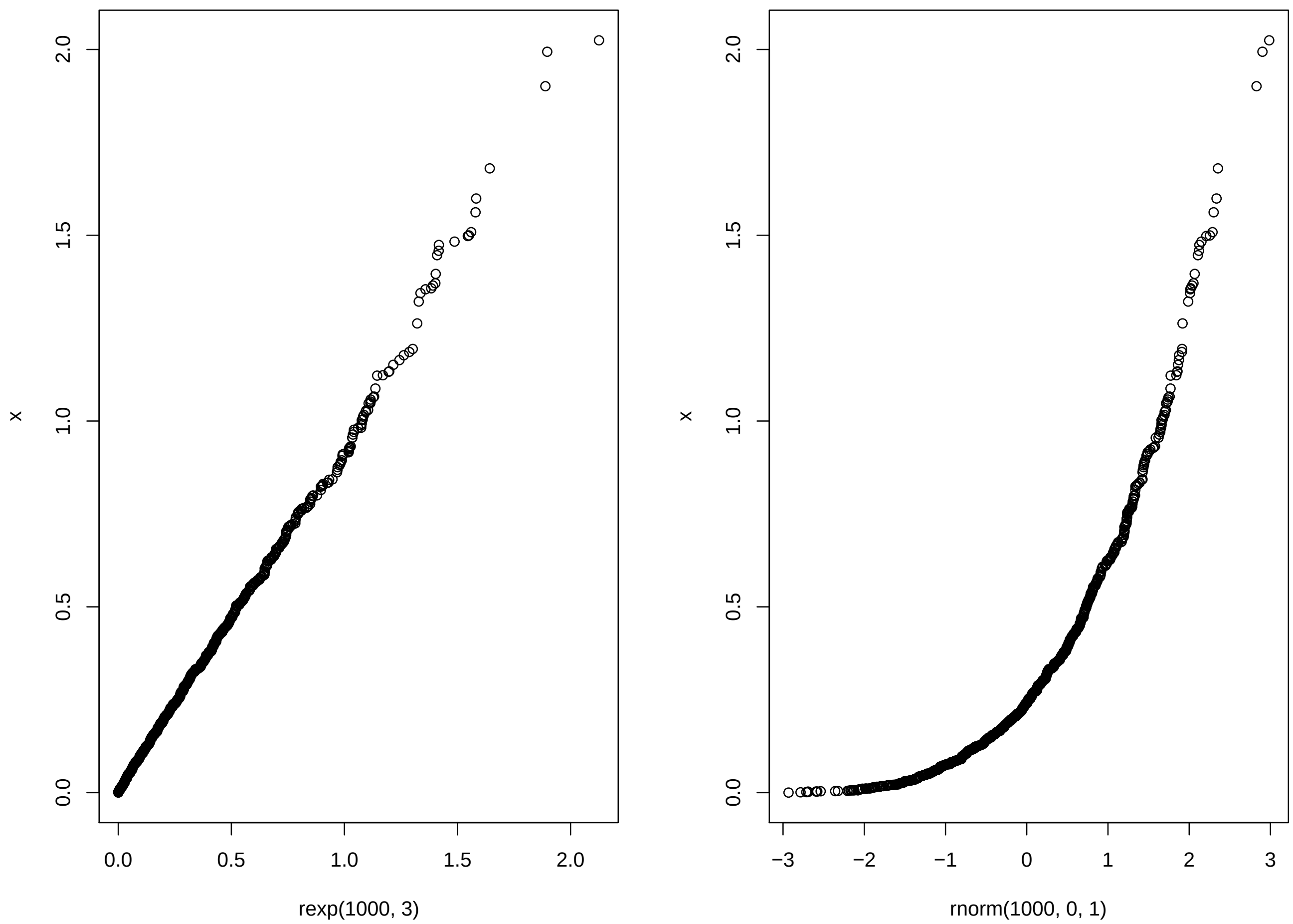

```
> qqnorm(galton,type="l")
```

QQ Plot



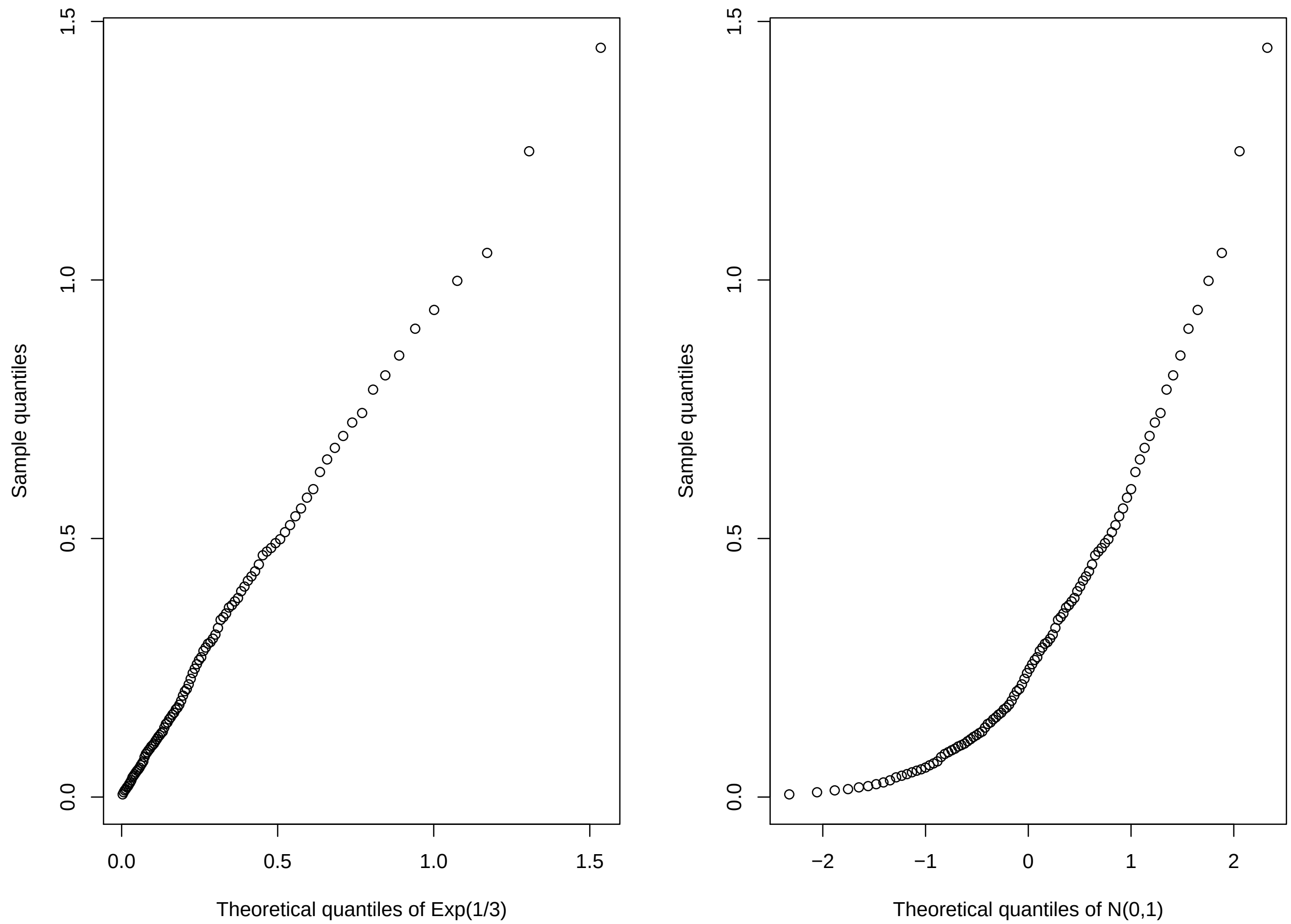

```
> par(mfcol=c(1,2)); qqnorm(rnorm(1000,0,1)); qqnorm(rexp(1000,3))
```

QQ Plot




```
> x <- rexp(1000,3)
> par(mfcol=c(1,2)); qqplot(rexp(1000,3),x); qqplot(rnorm(1000,0,1),x)
```

QQ Plot




```
> x <- rexp(1000,3); probs <- seq(.01,.99,length=100)
> qx <- quantile(x,probs); tqexp <- qexp(probs,3); tqnorm <- qnorm(probs,0,1)
> par(mfcol=c(1,2)); plot(tqexp,qx,xlab="Theoretical quantiles of Exp(1/3)",ylab="Sample quantiles")
> plot(tqnorm,qx,xlab="Theoretical quantiles of N(0,1)",ylab="Sample quantiles")
```

Some approximations

- The interval $\bar{x} \pm s$ will contain approx 68% of the observations
- The interval $\bar{x} \pm 2s$ will contain approx 95% of the observations
- Assuming $Y \sim N(\mu, \sigma^2)$

$$\Pr[\mu - \sigma < Y < \mu + \sigma] = \Pr[-1 < Z < 1] = 0.6827$$

$$\Pr[\mu - 2\sigma < Y < \mu + 2\sigma] = \Pr[-2 < Z < 2] = 0.9545$$

Some approximations

- Do these approximations hold for non-normal data?

  Not in general.

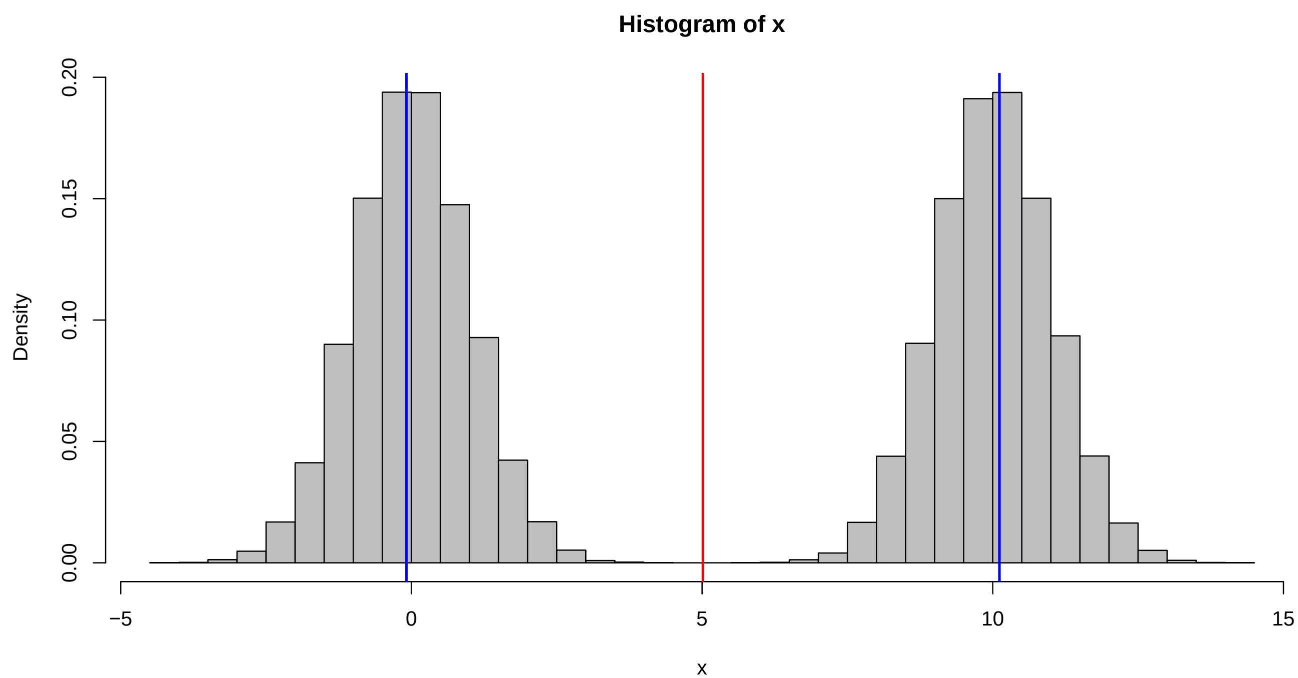
- Consider $X \sim Exp(\lambda)$ such that $EX = \lambda$ and $V(X) = \lambda^2$. For $\lambda = 1/3$,

$$\Pr[0 \leq X \leq 2/3] = 0.86$$

Some approximations

- Consider $X = WY + (1-W)Z$ where $W \sim Bern(1/2)$, $Y \sim N(10, 1)$, and $Z \sim N(0, 1)$. Can show

$$\Pr[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X] \approx 0.54$$



Some approximations

- The following holds for any rv Y with mean μ and variance σ^2

$$\Pr\left[\mu - K\sigma \leq Y \leq \mu + K\sigma\right] \geq 1 - \frac{1}{K^2}$$

  $\forall K \geq 1$. This is *Chebyshev's inequality* (note typo in text page 100)

- Eg, if $K = 2$,

$$\Pr\left[\mu - 2\sigma \leq Y \leq \mu + 2\sigma\right] \geq 0.75$$

  i.e. we would expect at least 75% of observations to be within two standard deviations of the mean *for any underlying distribution*

Central Limit Theorem (CLT)

- Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed (iid) random variables with

$$E(Y_i) = \mu \text{ and,}$$

$$Var(Y_i) = \sigma^2 > 0$$

- Define

$$Z_n = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}$$

- Then the distribution function of Z_n converges to a standard normal distribution function as $n \to \infty$.

Central Limit Theorem (CLT)

- In words - see Result 4.3 page 84 of van Belle et al
- If a random variable Y has population mean μ and population variance σ^2, then the sample mean $\bar{Y}$, based on n observations, is approx normally distributed with mean μ and variance σ^2/n for sufficiently large n

Notes on CLT

- The CLT applies to any distribution of the Y's
- The approximation improves as n gets large
- Check out Rice Virtual Lab in Statistics

http://onlinestatbook.com/rvls.html

Result 4.2

- If Y is normally distributed with mean μ and variance σ^2, then $\bar{Y}$, based on a random sample of n observations, is normally distributed with mean μ and variance σ^2/n.
- This is true regardless of sample size.