One Sample Tests
Bios 662

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Outline

- Small sample, Normal
- Large sample
- Nonparametric
  - Sign test
  - Wilcoxon signed rank test

Applications

- Prevalence study: collect data to test a hypothesis about mean or median of Y
- Paired data:

  Eg Before-after study where measure characteristic before and after treatment; twins

Small Sample, Normal

- For small sample and $Y \sim N(\mu, \sigma^2)$,
- Form previous lecture, test statistic

$$T = \frac{\bar{Y} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$$

- For two-sided alternative $H_A : \mu \neq \mu_0$, critical region

$$C_\alpha = \{t : |t| > t_{n-1,1-\alpha/2}\}$$

- For one-sided alternative $H_A : \mu > \mu_0$, critical region

$$C_\alpha = \{t : t > t_{n-1,1-\alpha}\}$$

Small Sample, Normal

- P-value: probability of obtaining test statistic as or more extreme than observed from the sample

- For two-sided alternative $H_A : \mu \neq \mu_0$,

$$p = \Pr[T \leq -|t|] + \Pr[T \geq |t|]$$

where $T \sim t_{n-1}$. Equivalently

$$p = 2\Pr[T \leq -|t|] = 2\Pr[T \geq |t|]$$

- For one-sided alternative $H_A : \mu > \mu_0$,

$$p = \Pr[T \geq t]$$

SIDS Example

- Example: text page 281; problem 8.2
- Investigators are interested in whether babies that die of SIDS are different in weight than babies who do not die of SIDS.
- A study of 22 dizygotic twins compared the birthweight of the baby who died with the baby who did not die.
- The data are given in the text.

SIDS Example

- Example: text page 281; problem 8.2
- Let our random variable Y equal the weight of SIDS baby (twin) minus weight of non-SIDS baby (twin)

$$H_0 : \mu_{diff} = 0$$

$$H_A : \mu_{diff} \neq 0$$

SIDS Example

- Critical region at $\alpha = 0.05$

$$C_{.05} = \{t : |t| > t_{21,.975} = 2.08\}$$

- $\bar{y} = 0.1818$, $s = 369.57$, $n = 22$

$$t = \frac{\bar{y}}{s/\sqrt{n}} = 0.0023$$

- P-value

$$p = 2 * \Pr[T \leq -0.0023] = 0.9982$$

Example: R

- R

```
> t.test(sid.diffs)

        One Sample t-test

data:  sid.diffs
t = 0.0023, df = 21, p-value = 0.9982
alternative hypothesis: true mean is not equal to 0


> t.test(sid.diffs,alternative="greater")

        One Sample t-test

data:  sid.diffs
t = 0.0023, df = 21, p-value = 0.4991
alternative hypothesis: true mean is greater than 0
```

Example: SAS

- SAS

```
Proc ttest; var diff;

                         T-Tests

         Variable      DF    t Value    Pr > |t|
         diff          21       0.00      0.9982
```

Small Sample

- t test assumptions:
  - Observations are independent
  - Sample is from normal distribution

Large Sample

- For large sample, use normal approximation (CLT)

$$\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$$

and Slutsky

$$Z = \frac{\bar{Y} - \mu}{s/\sqrt{n}} \text{ approx } N(0, 1)$$

- Approximation improves as $n \to \infty$
- Note: Y's do not need to be normally distributed

Large Sample

- Example: Iron deficiency
- Iron deficiency anemia is an important nutritional health issue.
- A study of 63 boys aged 9-11 from families with income below the poverty level was conducted.
- The mean daily iron intake in the U.S. population is known to be 14.45 mg.
- Q: Is iron deficiency in boys associated with family income?

Example: Iron deficiency (continued)

- Conduct large sample test

$$H_0 : \mu = 14.45; H_A : \mu \neq 14.45$$

$$C_{.05} = \{z : |z| > 1.96\}$$

$$\bar{y} = 12.5; s^2 = 22.5625; n = 63$$

$$z = \frac{12.5 - 14.45}{\sqrt{22.5626/63}} = -3.26$$

- Reject H_0. Q: What is $2\Phi(-3.26)$?

Testing/Estimation: Large Sample

- Testing $H_0 : \mu = \mu_0$ versus $H_A : \mu \neq \mu_0$

$$C_\alpha = \{z : |z| > z_{1-\alpha/2}\}$$

where

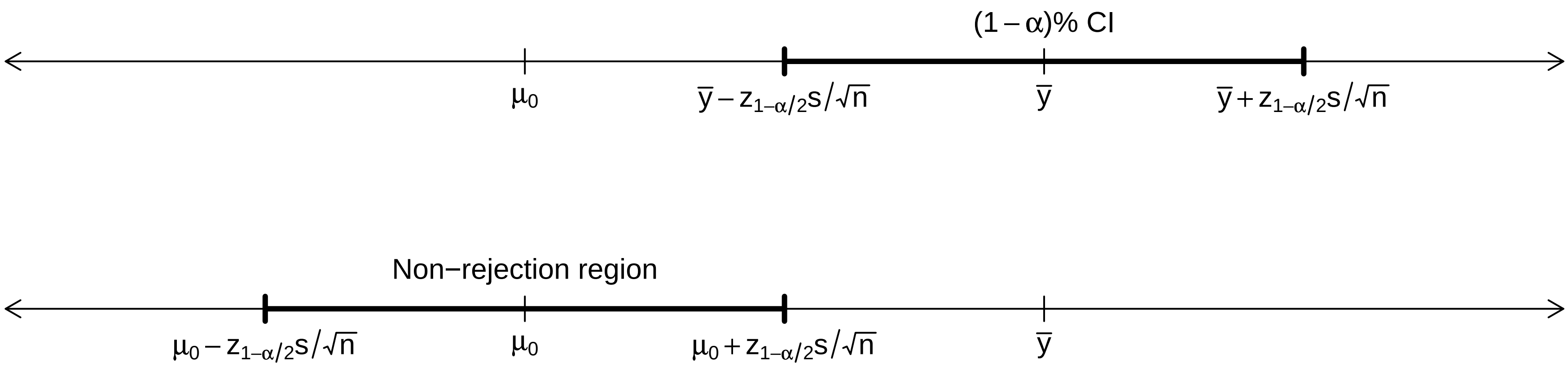
$$z = \frac{\bar{y} - \mu_0}{s/\sqrt{n}}$$

- Estimation: confidence interval for μ

$$\bar{y} \pm z_{1-\alpha/2} \frac{s}{\sqrt{n}}$$

- Can show: Reject H_0 iff CI excludes μ_0

Testing/Estimation: Large Sample



Testing/Estimation: Large Sample

- Theorem: Reject H_0 iff CI excludes μ_0
- Sketch of proof: Suppose CI excludes μ_0 ie

$$\mu_0 \notin [\bar{y} - z_{1-\alpha/2}\frac{s}{\sqrt{n}}, \bar{y} + z_{1-\alpha/2}\frac{s}{\sqrt{n}}]$$

Without loss of generality, assume

$$\mu_0 < \bar{y} - z_{1-\alpha/2}\frac{s}{\sqrt{n}},$$

This implies

$$z_{1-\alpha/2} < \frac{\bar{y} - \mu_0}{s/\sqrt{n}} \equiv z$$

implying $z \in C_\alpha$ $\Box$

Testing/Estimation

- Equivalent; two sides of the same coin
- See text Section 4.7
- Emphasize testing beforehand (to inform power, sample size calculations) and estimation afterward

Small Sample, Non-normal

- Transformation, bootstrap
- Nonparametric tests:
  - Sign test
  - Wilcoxon signed rank test
- Read van Belle et al 8.1-8.5

Sign Test

- Suppose $Y_1, \ldots, Y_n$ iid continuous F with median $\zeta_{.5}$
- Hypotheses

$$H_0 : \zeta_{.5} = \zeta_{.5,0} \text{ (median=some value)}$$

$$H_A : \zeta_{.5} \neq \zeta_{.5,0}$$

- If the true median is $\zeta_{.5,0}$ and F continuous,

$$\Pr[Y < \zeta_{.5,0}] = \Pr[Y > \zeta_{.5,0}] = .5$$

for a randomly selected observation Y

Sign Test

- Let R be the number of obs $> \zeta_{.5,0}$
- Under H_0

$$R \sim Bin(n, .5)$$

$$\Pr[R \leq r] = \sum_{i=0}^{r} \binom{n}{i} \left(\frac{1}{2}\right)^{i} \left(1 - \frac{1}{2}\right)^{n-i} = \frac{1}{2^n} \sum_{i=0}^{r} \binom{n}{i}$$

Sign Test

- Critical region

$$C_\alpha = \{r : r \le r_{\alpha/2} \text{ or } r \ge r_{1-\alpha/2}\}$$

where $r_{\alpha/2}$ and $r_{1-\alpha/2}$ are such that

$$\Pr[R \le r_{\alpha/2} | H_0] + \Pr[R \ge r_{1-\alpha/2} | H_0] \le \alpha$$

Sign Test

- Since Bin(n, .5) is symmetric, $r_{1-\alpha/2} = n - r_{\alpha/2}$
- Thus need $r_{\alpha/2}$ such that

$$\Pr[R \leq r_{\alpha/2} | H_0] = \frac{1}{2^n} \sum_{i=0}^{r_{\alpha/2}} \binom{n}{i} \leq \frac{\alpha}{2}$$

- Choose largest $r_{\alpha/2}$ such that this inequality holds; why?
- For $n = 10$, the critical region for a 2-sided test with $\alpha = 0.05$ is $C_{.05} = \{0, 1, 9, 10\}$

Sign Test

- CDF for Binomial($n = 10, \pi = .5$)

r	Cumulative Probability
0	.0010
1	.0107
2	.0547
3	.1719
4	.3770
5	.6231
6	.8282
7	.9453
8	.9893
9	.9990
10	1.0000

Sign Test: Example

Example: Calcium supplementation in African-American men

	treatment	before	after	diff
1.	calcium	107	100	-7
2.	calcium	110	114	4
3.	calcium	123	105	-18
4.	calcium	129	112	-17
5.	calcium	112	115	3
6.	calcium	111	116	5
7.	calcium	107	106	-1
8.	calcium	112	102	-10
9.	calcium	136	125	-11
10.	calcium	102	104	2

Sign Test: Example

- Testing

$$H_0 : \zeta_{.5} = 0 \text{ versus } H_A : \zeta_{.5} \neq 0$$

- For $n = 10$ and $\alpha = 0.05$, $C_{.05} = \{0, 1, 9, 10\}$
- $r = 4$ is not in $C_{.05}$; do not reject H_0
- P-value

$$2 \times \left\{ \frac{1}{2^{10}} \sum_{i=0}^{4} \binom{10}{i} \right\} = 0.754$$

Sign Test: Example

- R code

```
> 2*sum(dbinom(0:4,10,.5))
[1] 0.7539063

> 2*pbinom(4,10,.5)
[1] 0.7539063

> library("BSDA")
> sign.test(diff)

        One-sample Sign-Test

data:  diff
s = 4, p-value = 0.7539
alternative hypothesis: true median is not equal to 0
```

Sign Test: Example

- SAS Proc univariate output:

```
                    The UNIVARIATE Procedure
                         Variable:  diff

                             Moments

N                          10    Sum Weights                 10
Mean                       -5    Sum Observations           -50
Std Deviation      8.74325137    Variance            76.4444444
Skewness           -0.3378852    Kurtosis            -1.5550482
Uncorrected SS            938    Corrected SS               688
Coeff Variation    -174.86503    Std Error Mean      2.76485885

                   Tests for Location: Mu0=0

        Test           -Statistic-    -----p Value------

        Student's t    t  -1.80841    Pr > |t|    0.1040
        Sign           M        -1    Pr >= |M|   0.7539
        Signed Rank    S     -13.5    Pr >= |S|   0.1934
```

Sign Test: Comments

- Typically $\zeta_{.5,0} = 0$
- Alternative formulation of the null

$$\Pr[Y < \zeta_{.5,0}] = \Pr[Y > \zeta_{.5,0}]$$

- Eg if comparing difference in outcome under drug versus control, null says prob drug better than control is the same as the prob drug worse than control
- Delete obs=$\zeta_{.5,0}$ and reduce n by 1 for each

Sign Test: Large Samples

- If n is large, we can use the CLT for the sign test.
- Recall for $R \sim Bin(n, \pi)$

$$E(R) = n\pi \ , Var(R) = n\pi(1-\pi)$$

- Thus

$$Z = \frac{R - n\pi}{\sqrt{n\pi(1-\pi)}}$$

  will be appox $N(0, 1)$

- The approx gets better as $n \rightarrow \infty$

Sign Test: Large Samples

- For sign test, $H_0 : \pi = .5$
- Therefore we compute

$$Z = \frac{R - n/2}{\sqrt{n/4}}$$

- Critical region comes from $\Phi(z)$

Sign Test: Large Samples

- Normal appox to sign test works well for $n \geq 40$
- For $40 \leq n \leq 100$, approx better using the following adjustment:

$$Z = \begin{cases} \frac{R-(n+1)/2}{\sqrt{n/4}} & \text{if } (R - n/2) > 1/2 \\ \\ \frac{R-n/2}{\sqrt{n/4}} & \text{if } |R - n/2| \leq 1/2 \\ \\ \frac{R-(n-1)/2}{\sqrt{n/4}} & \text{if } (R - n/2) < -1/2 \end{cases}$$

Sign Test: Example

- A study was conducted to compare an automated machine for measuring blood pressure with measures made by a nurse using a standard mercury machine
- 100 people had their blood pressure measured using both techniques
- We use sign($Y = BP_{auto} - BP_{nurse}$)

$$H_0 : \Pr[Y < 0] = \Pr[Y > 0]$$

vs

$$H_A : \Pr[Y < 0] \neq \Pr[Y > 0]$$

Sign Test: Example

- For the study:

$$r = 64$$

$\Rightarrow$

$$r - n/2 > 1/2$$

$\Rightarrow$

$$z = 13.5/5 = 2.7$$

- Reject H_0; conclude automated machine is more likely to give higher reading

Binomial Continuity Correction

- Suppose $R \sim Bin(n, \pi)$ and $X \sim N(n\pi, n\pi(1-\pi))$
- By CLT

$$\Pr[R \leq x] \approx \Pr[X \leq x]$$

- Continuity correction

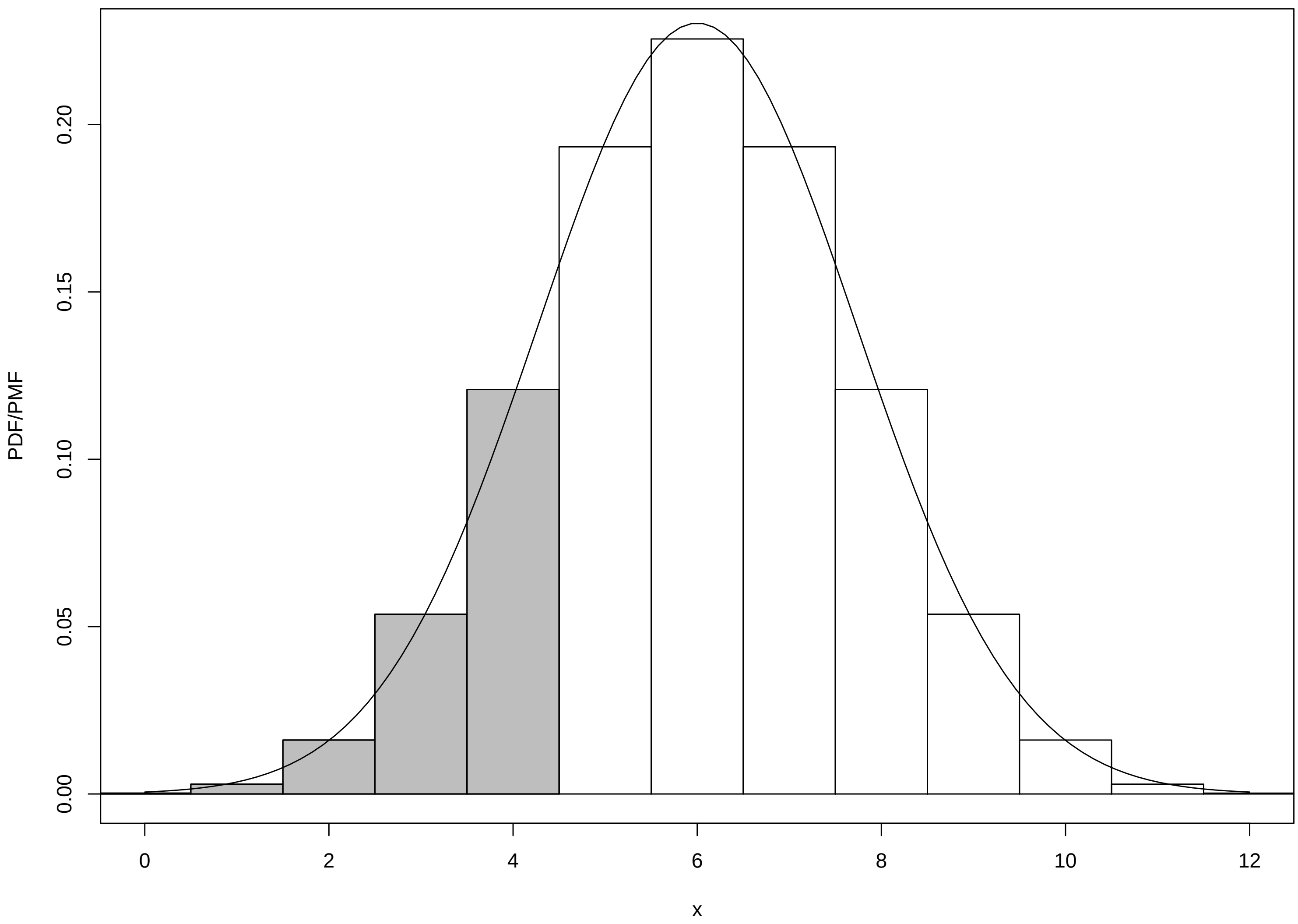
$$\Pr[R \leq x] \approx \Pr[X \leq x + 1/2]$$

- Eg $n = 12$, $\pi = 0.5$, $x = 4$

$$0.194 = \Pr[R \leq 4] \approx \Pr[X \leq 4 + 1/2] = 0.193$$

whereas $\Pr[X \leq 4] = 0.124$

Binomial Continuity Correction



Binomial Continuity Correction

- Likewise

$$\Pr[R \geq x] \approx \Pr[X \geq x - 1/2]$$

- Returning to sign test where $\pi = 1/2$, if $r < (n-1)/2$ then one-sided p-value will equal

$$\Pr[R \leq r] \approx \Pr[X \leq r{+}1/2] = \Pr\left[Z \leq \frac{r + 1/2 - n/2}{\sqrt{n/4}}\right]$$

Sign Test: Efficiency

- Def 8.6 The *relative efficiency* of statistical procedure A compared to B is the ratio of the sample size needed for B to that of A in order for both procedures to have the same statistical power
- If sampling from a normal distribution
  - n small: the sign test is almost as efficient as the t-test
  - As n gets large,the sign test becomes less efficient and the $ARE = 2/\pi = 0.64$
- If sampling from a non-normal distribution, the sign test can be more efficient

Wilcoxon Signed Rank Test

- Suppose $Y_1, Y_2, \ldots, Y_n$ iid according to a symmetric distribution F with median $\zeta_{.5}$
- Hypotheses

$$H_0 : \zeta_{.5} = \zeta_{.5,0}$$

vs

$$H_A : \zeta_{.5} \neq \zeta_{.5,0}$$

Wilcoxon Signed Rank Test

- Delete Y_i's equal $\zeta_{.5,0}$, adjust n
- Compute $Y_i' = Y_i - \zeta_{.5,0}$
- Rank $|Y_i'|$'s from smallest to largest
- The statistic S^+ is the sum of ranks from observation with Y_i' positive
- S^- defined similarly

Wilcoxon Signed Rank Test: Example

Example: Calcium supplementation in African-American men

	treatment	before	after	diff	absol	rank	sgn*rank
1.	calcium	107	100	-7	7	6	-6
2.	calcium	110	114	4	4	4	4
3.	calcium	123	105	-18	18	10	-10
4.	calcium	129	112	-17	17	9	-9
5.	calcium	112	115	3	3	3	3
6.	calcium	111	116	5	5	5	5
7.	calcium	107	106	-1	1	1	-1
8.	calcium	112	102	-10	10	7	-7
9.	calcium	136	125	-11	11	8	-8
10.	calcium	102	104	2	2	2	2

Wilcoxon Signed Rank Test: Example

- $S^+ = 4 + 3 + 5 + 2 = 14$; $S^- = 41$
- $S = \min\{S^+, S^-\} = 14$
- Table on course website gives critical values
- For $n = 10$, $C_{0.05} = \{S : S \leq 8\}$
- Therefore, do not reject H_0 median is 0

Wilcoxon Signed Rank Test

- Since
$$\sum_{i=1}^{n} i = n\left(\frac{n+1}{2}\right)$$
- It follows
$$S^{+} + S^{-} = n\left(\frac{n+1}{2}\right)$$
- So only smaller values are tabulated

Wilcoxon Signed Rank Test

- How to compute null distribution of signed-rank test?
- Under the null, each ranked obs has prob 1/2 of having pos sign
- The n signs are independent
- There are 2^n possible outcomes
- Thus each outcome occurs w/ prob $1/2^n$

Distribution of S^+ under H_0

- Calculating the null distribution for $n = 4$; an x in the column indicates that the sign of the rank is positive

1	2	3	4	S^+
				0
x				1
	x			2
		x		3
			x	4
x	x			3
x		x		4
x			x	5
	x	x		5
	x		x	6
		x	x	7
x	x	x		6
x	x		x	7
x		x	x	8
	x	x	x	9
x	x	x	x	10

Distribution of S^+ under H_0

k	$\Pr[S^+ \leq k]$
0	1/16=0.0625
1	1/8=0.125
2	3/16=0.1875
3	5/16=0.3125
4	7/16=0.4375
5	9/16=0.5625
6	11/16=0.6875
7	13/16=0.8125
8	7/8=0.8750
9	15/16=0.9375
10	1

Table A.9: Distribution of S^+ under H_0

n	k	$\Pr[S^+ \le k]$
4	0	1/16=0.0625
	1	1/8=0.125
	$\vdots$	$\vdots$
6	0	$1/2^6$=0.015625
	1	1/32=0.03125
	$\vdots$	$\vdots$
7	0	$1/2^7$=0.0078125
	1	$2/2^7$=0.015625
	2	$3/2^7 = 0.0234$
	3	$5/2^7 = 0.039$

* Bold face values denote critical values listed in Table A.9 of text page 834 for one sided $\alpha = 0.025$ and two-sided $\alpha = 0.05$. Is cv in critical region?

Distribution of S^+ under H_0

- Large sample distribution
- Can show

$$E(S^+) = \frac{n(n+1)}{4} \text{ and } V(S^+) = \frac{n(n+1)(2n+1)}{24}$$

- If $n \geq 15$,

$$Z = \frac{S^+ - n(n+1)/4}{\sqrt{n(n+1)(2n+1)/24}} \sim N(0,1)$$

Wilcoxon Signed Rank Test: Ties

- If there are 2 or more observations with the same value of Y', the observations are said to be *tied*
- For tied observations we assign the average rank or *midrank*
- Example: $\boldsymbol{Y} = \{23, 25, 45, 13, 23, 46\}$

  MidRanks: $\{2.5, 4, 5, 1, 2.5, 6\}$

Wilcoxon Signed Rank Test: Ties

- Can show
$$E(S^+) = \frac{n(n+1)}{4}$$
- To accommodate ties, var is adjusted
$$V(S^+) = \frac{n(n+1)(2n+1) - \frac{1}{2}\sum_{i=1}^{q} t_i(t_i-1)(t_i+1)}{24}$$
where q equals the number of sets of ties and t_i is the number of observations in the ith set
- For example on previous slide, $q = 1$ and $t_1 = 2$ such that
$$V(S^+) = \frac{6(6+1)(2\cdot 6+1) - \frac{1}{2}\cdot 2\cdot 1\cdot 3}{24}$$

Wilcoxon Signed Rank Test: Ties

- If n is large, use the variance adjusted for ties in the normal approximation
- If n is small and there are ties, need to compute the null distribution from permutation principles, i.e., tables of critical values are not guaranteed to be correct in the presence of ties

Wilcoxon Signed Rank Test: Example w/ Ties

- From Table 8.7, page 281

	SIDS	nonSIDS	Y	Y	rank	sgnrnk
1.	1474	2098	-624	624	21	-21
2.	3657	3119	538	538	19	19
3.	3005	3515	510	510	18	18
4.	2041	2126	-85	85	3	-3
5.	2325	2211	114	114	4	4
6.	2296	2750	-454	454	15	-15
7.	3430	3402	28	28	1	1
8.	3515	3232	283	283	9	9
9.	1956	1701	255	255	7	7
10.	2098	2410	-312	312	11	-11
11.	3204	2892	312	312	11	11
12.	2381	2608	-277	277	8	-8
13.	2892	2693	199	199	6	6
14.	2920	3232	-312	312	11	-11
15.	3005	3005	0	0		
16.	2268	2325	-57	57	2	-2
17.	3260	3686	-426	426	14	-14
18.	3260	2778	482	482	16.5	16.5
19.	2155	2552	-397	397	13	-13
20.	2835	2693	142	142	5	5
21.	2466	1899	567	567	20	20
22.	3232	3714	-482	482	16.5	-16.5

Wilcoxon Signed Rank Test: Example w/ Ties

- $S^{+} = 116.5$; $E(S^{+}) = 21(22)/4 = 115.5$
- $q = 2$; $t_1 = 3$; $t_2 = 2$ such that

$$V(S^{+}) = \frac{21(22)(43) - \frac{1}{2}\{3(2)(4) + 2(1)(3)\}}{24} = 827.15$$

- Thus

$$Z = \frac{116.5 - 115.5}{\sqrt{827.15}} = 0.03477$$

- Yielding $p = \Phi(-0.03477) * 2 = 0.9723$

Wilcoxon Signed Rank Test: Example w/ Ties

- R code:

```
> wilcox.test(sid.diffs,exact=F,correct=F)

        Wilcoxon signed rank test

data:  sid.diffs
V = 116.5, p-value = 0.9723
alternative hypothesis: true mu is not equal to 0
```

- SAS uses slightly different large sample approximation; Proc Univariate yields $p = 0.9733$

Efficiency of Signed Rank Test

- If the Y's come from a normal distribution, the ARE of the Signed Rank test compared to the normal distribution test is $3/\pi = 0.955$

  (cf Lehmann 1998, page 80)

- Gain in efficiency over sign test due to additional assumption: symmetry

Signed rank test: H_A

- Suppose $n = 10$, $\alpha = .05$, $H_0 : \zeta_{.5} = 0$
- For $H_A : \zeta_{.5} \neq 0$, $S = \min\{S^+, S^-\}$

$$C_{0.05} = \{s : s \leq 8\}$$

- Equivalently

$$C_{0.05} = \{s^+ : s^+ \leq 8 \text{ or } s^+ \geq 47\}$$

or

$$C_{0.05} = \{s^- : s^- \leq 8 \text{ or } s^- \geq 47\}$$

Signed rank test: H_A

- For $H_A : \zeta_{.5} < 0$, what is rejection region?

$$C_{0.05} = \{s^+ : s^+ \leq 10\}$$

or

$$C_{0.05} = \{s^- : s^- \geq 45\}$$