Analysis of Variance III
Bios 662

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Outline

- Diagnostics
- Nonparameteric Alternative: Kruskal-Wallis

ANOVA: Diagnostics

- Diagnostics discussed in section 10.6 of text
- Assumptions

  1. Homogeneity of variance
  2. Normality of residual error
  3. Independence of residual error
  4. Linearity

ANOVA: Diagnostics

- Homogeneity of variance
  - Inspect plot of raw data or standard deviations by group means
  - Hartley's and Cochran's test

    $$F_{MAX} = \frac{s_{max}^2}{s_{min}^2}, \ C = \frac{s_{max}^2}{\sum s_i^2}$$

    tables given in Web appendix of text
  - These tests require equal sample size and sensitive to normality assumption

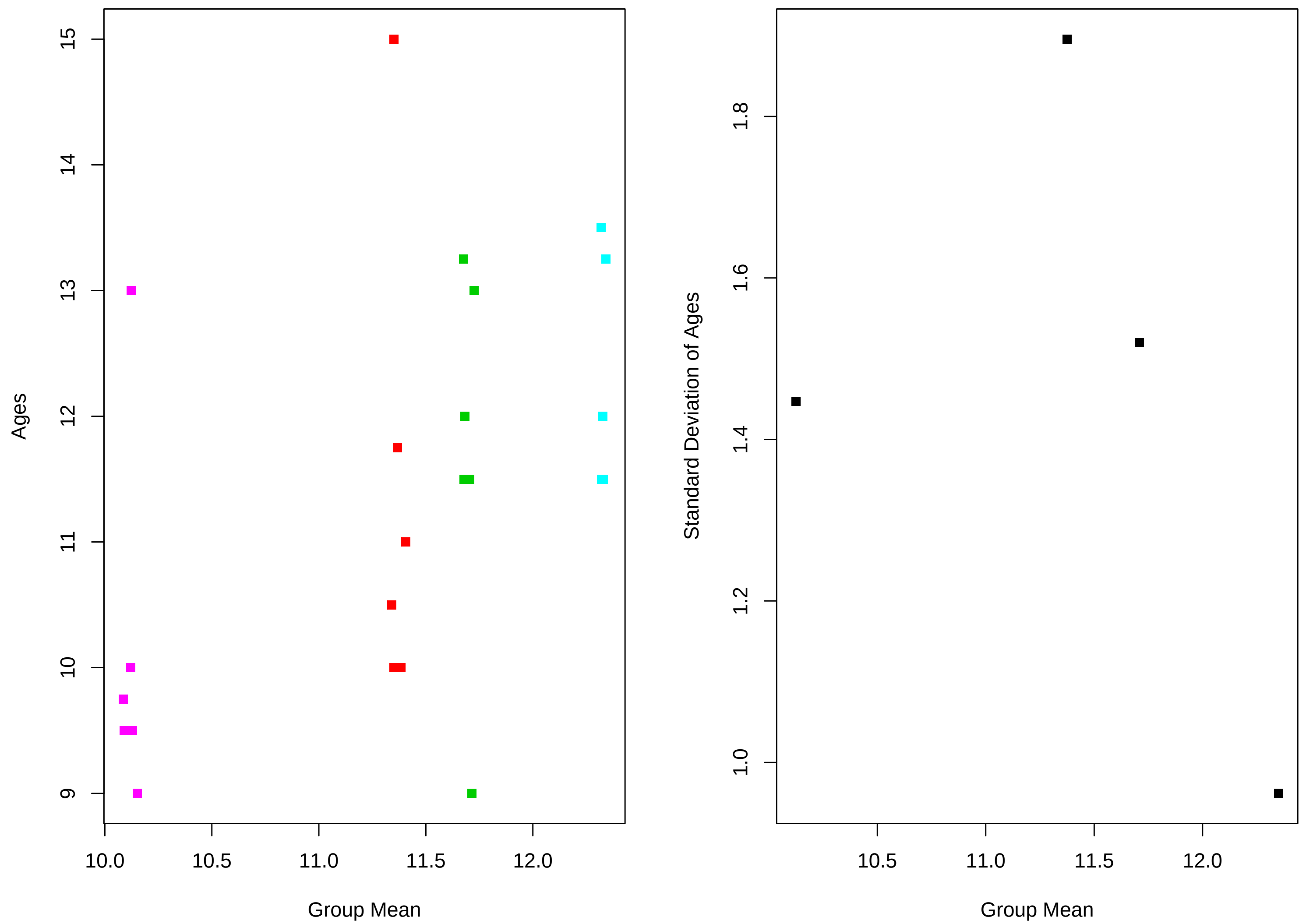
ANOVA: Diagnostics

- Homogeneity of variance
  - Modified-Levene test (Brown-Forsythe): apply ANOVA to abs dev from group medians

$$d_{ij} = |Y_{ij} - \tilde{Y}_{i\cdot}|$$

    use usual F test; rejection indicates lack of homogeneity
  - Robust to normality, does not require equal sample sizes
  - Cf Chapter 18.2 of Kutner et al. *Applied Linear Statistical Models*, 5th Edition, 2005

Homogeneity of Variance Plot



Modified Levene Test: SAS

```
proc anova;  class group;  model age=group;  means group/hovtest=bf;
```

The ANOVA Procedure

Brown and Forsythe's Test for Homogeneity of age Variance
ANOVA of Absolute Deviations from Group Medians

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
group	3	0.8003	0.2668	0.19	0.9001
Error	19	26.3125	1.3849		

Level of group	N	age Mean	age Std Dev
active	6	10.1250000	1.44697961
eight	5	12.3500000	0.96176920
no	6	11.7083333	1.52000548
passive	6	11.3750000	1.89571886

Modified Levene Test: R

```
Levene <- function(y, group)
{
    group <- as.factor(group)  # precautionary
    meds <- tapply(y, group, median)
    resp <- abs(y - meds[group])
    anova(lm(resp ~ group))[1, 4:5]
}

> Levene(age,group)
      F value Pr(>F)
group  0.1926 0.9001
```

ANOVA: Diagnostics for Normality

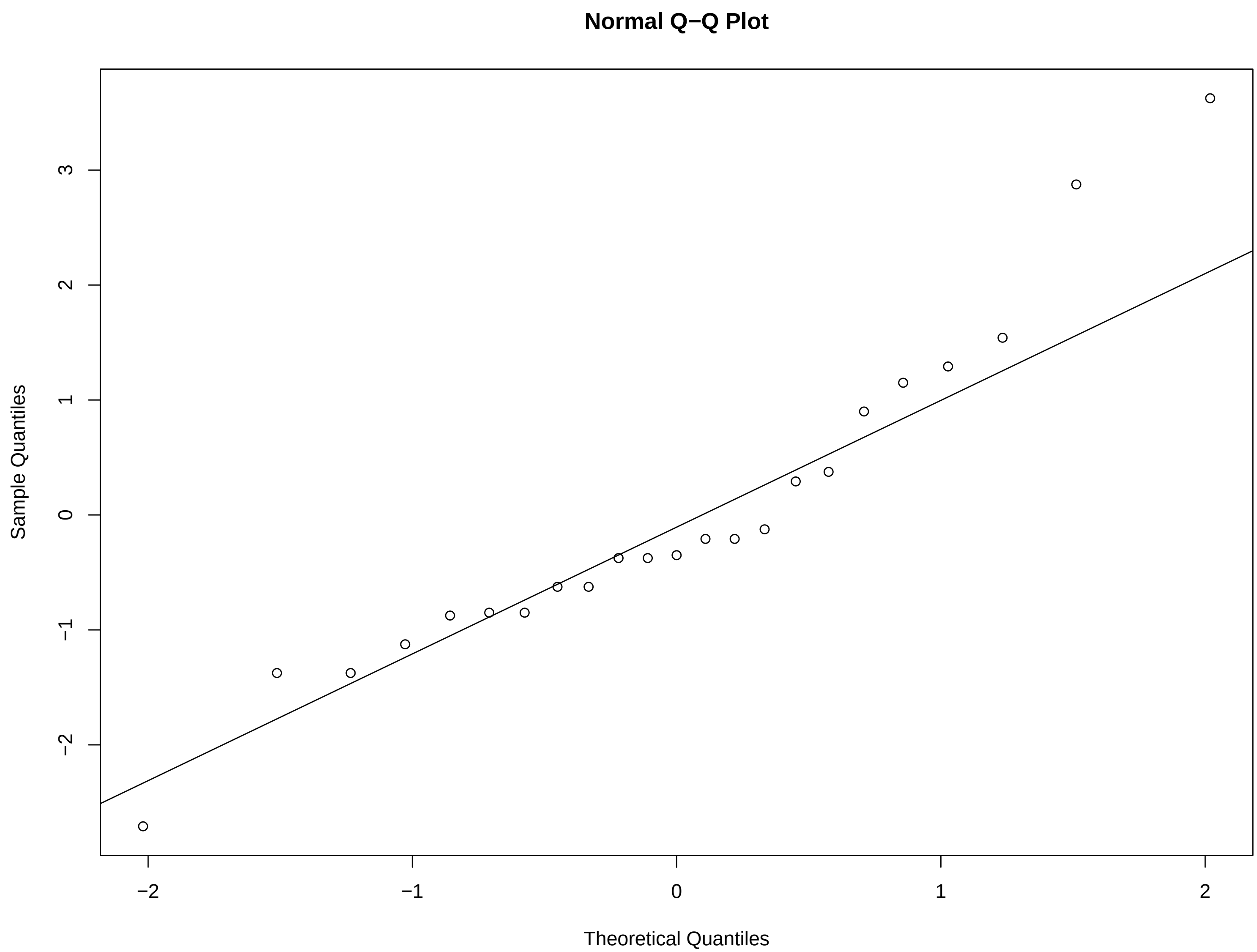
- QQ plot
- K-S GOF test
- Pearson correlation coefficient test:
  - Ordered residuals and expected values under normality
  - Assumption of normality in question if observed correlation is less than critical value on the next slide

ANOVA: Diagnostics for Normality

- Critical values for $\alpha = 0.05$

N	CV	N	CV	N	CV
5	0.88	10	0.92	24	0.96
6	0.89	12	0.93	30	0.96
7	0.90	15	0.94	40	0.97
8	0.91	20	0.95	50	0.98
9	0.91	22	0.95	100	0.99

ANOVA: Diagnostics for Normality



ANOVA: Diagnostics for Normality in R

```
> av <- aov(age ~ group)
> qq <- qqnorm(av$residuals)
> qqline(av$residuals)
> cor.test(qq$x,qq$y)

        Pearson's product-moment correlation

data:  qq$x and qq$y
t = 15.7572, df = 21, p-value = 4.146e-13
alternative hypothesis: true correlation is not equal to
0
95 percent confidence interval:
 0.9070150 0.9832468
sample estimates:
      cor
0.9602173
```

ANOVA: Diagnostics

- Remedial measures
  1. Normality: appeal to CLT
  2. Transformations
     - Plot $(\bar{y}_{i\cdot}, s_i)$, $(\bar{y}_{i\cdot}, s_i^2)$, $(\bar{y}_{i\cdot}^2, s_i)$; linearity suggests $\log(y)$, $\sqrt{y}$, $1/y$ transformations
     - Box-Cox family: minimize SSE (ie within group SS)
  3. Nonparametrics, eg, Kruskal-Wallis

Box-Cox Transformations

- Family of transformations indexed by λ

$$Y_\lambda = \begin{cases} k_1(Y^\lambda - 1) & \text{for } \lambda \neq 0 \\ k_2 \log(Y) & \text{for } \lambda = 0 \end{cases}$$

where

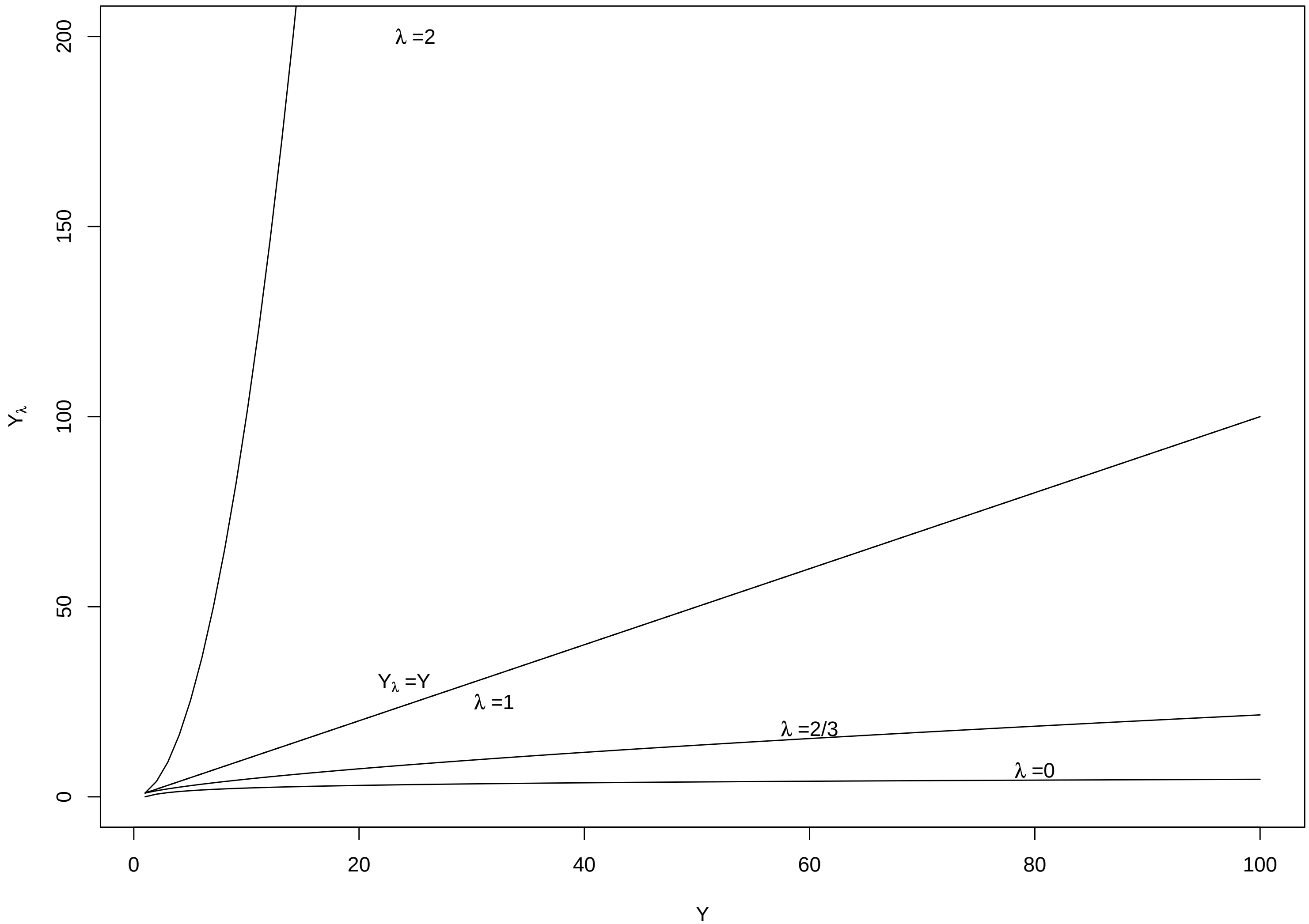
$$k_2 = \left(\prod_{i,j} Y_{ij} \right)^{1/N} \quad \text{and } k_1 = \frac{1}{\lambda k_2^{\lambda-1}}$$

- Choose λ that minimizes SSW

- SAS: macro on course website

  R: MASS library, function boxcox()

Box-Cox Transformations



Kruskal-Wallis

- Assume
$$Y_{ij} = \mu_i + \epsilon_{ij}$$
for $i = 1, \ldots, K; j = 1, \ldots, n_i$.
- ϵ_{ij} are independent and identically distributed with mean zero, but not necessarily normal

Kruskal-Wallis

- Same hypotheses

$$H_0 : \mu_1 = \cdots = \mu_K \text{ vs } H_A : \text{ at least one } \neq$$

- Pool all N observations and rank from smallest to largest
- Let R_{ij} be the rank of the j^{th} obs in the i^{th} group
- Let $\bar{R}_i = \sum_{j=1}^{n_i} R_{ij}/n_i$ equal the average rank in the i^{th} group
- Let $\bar{R}$ denote the overall average rank. What must this equal?

Kruskal-Wallis

- The Kruskal-Wallis test statistic

$$T_{KW} = \frac{12 \sum_{i=1}^{K} n_i (\bar{R}_i - \bar{R})^2}{N(N+1)}$$

- Equivalently

$$T_{KW} = \frac{12 \sum_{i=1}^{K} (\sum_{j=1}^{n_i} R_{ij})^2 / n_i}{N(N+1)} - 3(N+1)$$

- Reject H_0 for large values of T_{KW}

Kruskal-Wallis

- Under H_0, if the n_i are moderately large (rule of thumb: $n_i \geq 5$), then
$$T_{KW} \sim \chi^2_{K-1}$$
- If the n_i are small, the exact distribution of T_{KW} can be computed

Kruskal-Wallis: Exact

- There are
$$\binom{N}{n_1 n_2 \cdots n_K} = \frac{N!}{n_1! n_2! n_3! \cdots n_K!}$$
possible ways to assign n_1 ranks to group 1, n_2 ranks to group 2, ...

- Under H_0 each occurs with equal probability

- Suppose $n_1 = 2$, $n_2 = n_3 = 1$. Then
$$\binom{N}{n_1 n_2 \cdots n_K} = \frac{4!}{2!1!1!} = 12$$

Kruskal-Wallis: Exact

R_{1j}	R_{2j}	R_{3j}	$\sum_i R_{i\cdot}^2/n_i$	T_{KW}
1 2	3	4	9/2+9+16=29.5	2.7
1 3	2	4	28	1.8
1 4	2	3	25.5	0.3
2 3	1	4	29.5	2.7
2 4	1	3	28	1.8
3 4	1	2	29.5	2.7

k	$\Pr[T_{KW} = k]$
0.3	1/6
1.8	1/3
2.7	1/2

Kruskal-Wallis with Ties

- If there are ties among the ranks, we use the midrank method as in the Wilcoxon tests
- The KW statistic adjusted for ties is:

$$T_{KWadj} = \frac{T_{KW}}{1 - \sum_{i=1}^{q}(t_i^3 - t_i)/(N^3 - N)}$$

  where q is the number of sets of tied observations and t_i is the number of observations in the i^{th} set

- T_{KWadj} will also be approximately χ^2_{K-1}

Kruskal-Wallis: Example

- A study was conducted to compared three doses of aspirin in the treatment of fever in children with the flu
- 15 children with a fever between 100.0 and 100.9 F were randomly assigned to each dose ($n_1 = n_2 = n_3 = 5$, $N = 15$)
- Temperature was measured three hours later
- $H_0 : \mu_1 = \mu_2 = \mu_3$

Kruskal-Wallis: Example

- Distribution of T_{KW} (Owen 1962, page 422; KW 1953 Table 6.1)

k	$\Pr[T_{KW} \geq k]$
4.50	0.102
4.56	0.100
5.66	0.051
5.78	0.049
7.98	0.010
8.00	0.009

- $C_{.05} = \{T_{KW} \geq 5.78\}$

Kruskal-Wallis: Example

Low		Med		High	
T	R	T	R	T	R
2.0	14	0.6	8	1.1	10
1.6	13	1.2	11	-1.0	1
2.1	15	0.5	7	-0.2	3
0.7	9	0.2	4	0.4	6
1.3	12	-0.4	2	0.3	5

- $R_1 = 63,\ R_2 = 32,\ R_3 = 25$

Kruskal-Wallis: Example

- Therefore

$$T_{KW} = \frac{12(63^2/5 + 32^2/5 + 25^2/5)}{15(16)} - 3(16) = 8.18$$

- Asymptotic p-value

$$\Pr[\chi^2_2 > 8.18] = 0.0167$$

- From Owen table, expect exact p-value < 0.009

Kruskal-Wallis: SAS

```
proc npar1way;  class dose;  var change;  exact wilcoxon;
```

Wilcoxon Scores (Rank Sums) for Variable change
Classified by Variable dose

dose	N	Sum of Scores	Expected Under H0	Std Dev Under H0	Mean Score
low	5	63.0	40.0	8.164966	12.60
med	5	32.0	40.0	8.164966	6.40
high	5	25.0	40.0	8.164966	5.00

Kruskal-Wallis Test

Chi-Square	8.1800
DF	2
Asymptotic Pr > Chi-Square	0.0167
Exact Pr >= Chi-Square	0.0081

Kruskal-Wallis: R

```
> kruskal.test(change,dose)

        Kruskal-Wallis rank sum test

data:  change and dose
Kruskal-Wallis chi-squared = 8.18, df = 2, p-value = 0.01674
```

Kruskal-Wallis

- Suppose we perform ANOVA w/ Y_{ij}'s replaced by their ranks
- Resulting F test

$$F_R = \frac{(N-K)T_{KW}}{(K-1)(N-1-T_{KW})}$$

Kruskal-Wallis

- If $K = 2$, KW test equivalent to the Wilcoxon ranksum test
- ARE is $3/\pi = 0.955$ compared to F-test under normality
- For multiple comparisons of means, use Wilcoxon ranksum tests with Bonferroni correction