Solutions to Problem Set 11

19.5.3 $\mathcal{H}_\infty 2 = \{h_2 \in D[0,\tau] : \text{function}\, h_2\, \text{with bounded total variation}\}$;$\sigma_\theta^{22}$ is a linear operator from $\mathcal{H}_\infty 2$ to $\mathcal{H}_\infty 2$, and

$$\sigma_\theta^{22} = \frac{(1+\delta)e^{\beta' Z}Y(s)h_2(s)}{1+e^{\beta' Z}A(u)} - \hat{\xi}_\theta e^{\beta' Z}\int_0^\tau Y(u)h_2(u)dA(u)Y(s) \qquad (1)$$

Define $g(s) = P\left[\frac{(1+\delta)e^{\beta_0' Z}Y(s)h_2(s)}{1+e^{\beta_0' Z}A_0(u)}\right]$,$\sigma_\theta^{22}(h_2)(s) = g(s)h_2 + \sigma_2 h_2$,then σ_2 is a compact operator, i.e the range of $h \mapsto \sigma_2(h)$ over the unit ball in $H_\infty 2$ lies within a compact set. This is because the second term of (1) is bounded ranging over the unit ball. For any $g_2 \in \mathcal{H}_\infty 2$, there exists $h_2 = g_2(\cdot)/g(\cdot) \in \mathcal{H}_\infty 2$, then $g(\cdot)h_2 = g_2$. Thus $g(\cdot)$ is onto. It is also true that

$$\|g(\cdot)h_2(\cdot)\|_{\mathcal{H}_\infty 2} \geq \left(\inf_{s\in[0,\tau]} |g(s)|\right)\|h_2\|_{\mathcal{H}_\infty 2} \geq c_0\|h_2\|_{\mathcal{H}_\infty 2}.$$

Thus $g(\cdot)$ is both continuously invertible and onto.

If we can also verify that σ_θ^{22} is one-to-one, we then have by lemma 6.17 that σ_θ^{22} is both continuously invertible and onto. For any $h \in \mathcal{H}_\infty 2$, $\sigma_\theta^{22}(h) = 0$, we will prove $h = 0$. Define $t \to \theta \circ t = A_0 + \int_0^{(\cdot)} h(s)dA(s)$. $\sigma_{\theta_0}^{22} = 0 \Rightarrow P[-\frac{\partial 2}{\partial t2}\ln(\theta \circ t)|_{t=0}] = P[V^\tau(\theta_0)(h)]2 = 0$. $V^u(\theta_0)(h)$ is a continuous time martingale over $u \in [0,\tau]$.

$$V^u(\theta_0)(h) = \int_0^u \left(\frac{\dot{\lambda}_{\theta_0}(s)}{\lambda_{\theta_0}(s)}\right)(h_1' Z + h_2(s))dM(s),$$

where $\dot{\lambda}_{\theta_0} = \frac{\partial}{\partial t}\lambda_{\theta \circ t}|_{t=0}$, $\lambda_\theta(u) = \frac{e^{\beta' Z}a_0(u)}{1+e^{\beta' Z}A(u)}$, and $M(u) = N(u) - \int_0^u Y(s)\lambda_{\theta_0}(s)ds$ is a martingale.

Therefore,

$$\begin{aligned}
& P[V^\tau(\theta_0)(h)]2 = P[V^u(\theta_0)(h)]2 + P[V^\tau(\theta_0)(h) - V^u(\theta_0)(h)]2, \forall u \in [0,\tau] \\
\Rightarrow\ & P[V^u(\theta_0)(h)]2 = 0, \forall u \in [0,\tau] \\
\Rightarrow\ & V^u(\theta_0)(h) = 0, \text{ a.s} \,\forall u \in [0,\tau] \\
\Rightarrow\ & \frac{e^{\beta_0' Z}\int_0^u h(s)Y(s)dA_0(s)}{1+\int_0^u e^{\beta_0' Z}Y(s)dA_0(s)} = 0, \forall u \in [0,\tau] \\
\Rightarrow\ & \int_0^u h(s)Y(s)dA_0(s) = 0, \text{ a.s} \,\forall u \in [0,\tau] \\
\Rightarrow\ & \frac{\partial}{\partial u}\int_0^u h(s)Y(s)dA_0(s) = h(u)Y(u) = 0, \text{ a.s} \,\forall u \in [0,\tau] \\
\Rightarrow\ & h(u) = 0, \text{ a.s} \,\forall u \in [0,\tau]
\end{aligned}$$

Therefore, $\sigma^{22}_{\theta_0}$ is one-to-one since h is arbitary satisfying $\sigma^{22}_{\theta_0} = 0$.The desired result follows.

19.5.7 i) Verify $\mathcal{F}_1 = \{\dot{l}(t,\beta,\lambda) : (t,\beta,\lambda) \in V\}$ is a P_0-Donsker.

$Z \in \mathbb{R}^k$ is a regression covariate, restricted in a compact set of $\mathbb{R}^k$, thus $\{Z\delta\}, \{Z\}, \{h_0(w)\delta\}$ and $\{h_0(w)\}$ are all bounded Donsker class.Therefore, $\{e^{t'Z} : \|t - \beta_0\| < \epsilon\}$ is a bounded Donsker, since $f(x) = e^x$ is a lipschiz continuous function.

Similarly, $\{(\beta - t)'h_0(w), \|\beta - \beta_0\| < y, \|t - \beta_0\| < \epsilon\}$ and

$\left\{\frac{1}{1+(\beta-t)'h_0(w)}, \|\beta - \beta_0\| < y, \|t - \beta_0\| < \epsilon\right\}$ are bounded Donskers.

Thus,$\left\{\frac{h_0(w)\delta}{1+(\beta-t)'h_0(w)}, \|\beta - \beta_0\| < y, \|t - \beta_0\| < \epsilon\right\}$ is a Donsker.

Since $\mathcal{G} \equiv \{\Lambda_t(\beta,\Lambda) : (t,\beta,\Lambda) \in V\}$ is a subset of the class of all monotone functions $f : [0,\tau] \mapsto [0,M]$, which is known to be Donsker for any probability measure. Since $g \mapsto g(w), g \in \mathcal{G}$ is Lipschitz continuous, then $\{\Lambda_t(\beta,\Lambda)(w), (t,\beta,\Lambda) \in V\}$ is a Donsker. Also, it is bounded. Therefore, $\{Ze^{t'Z}\Lambda_t(\beta,\Lambda)(w), (t,\beta,\Lambda) \in$

$V\}$ is a Donsker.

$\left\{\int_0^{(\cdot)} h_0(s)d\Lambda(s), \|\Lambda - \Lambda_0\| < r\right\}$ has bounded variation over $[0,\tau]$.Notice that $s \mapsto h_0(s)$ is bounded in total variation. Then $\left\{\int_0^w h_0(s)d\Lambda(s) : \|\Lambda - \Lambda_0\| < r\right\}$ is bounded Donsker, and $\left\{e^{t'Z}\int_0^w h_0(s)d\Lambda(s), (t,\beta,\Lambda) \in V\right\}$ is a Donsker.

Therefore,$\{\dot{l}(t,\beta,\Lambda), (t,\beta,\Lambda) \in V\}$ is P_0-Donsker.

ii) Verify $\mathcal{F}_2, \ddot{l}(t,\beta,\Lambda) : (t,\beta,\Lambda) \in V$ is P_0-GC.Here,

$$\begin{aligned}\ddot{l}(t,\beta,\Lambda) = & - ZZ'e^{t'Z}\lambda_t(\beta,\Lambda)(w) + e^{t'Z}\int_0^w (Zh_0'(s) + h_0(s)Z')d\Lambda(s) \\ & - \frac{h_0(w)h_0'(w)s}{[1 + (\beta - t)'h_0(w)]2}\end{aligned}$$

From above argument, we have $e^{t'Z}\Lambda t(\beta,\Lambda)(w)$ is a Donsker. $h_0(s)$ is bounded in total variation, then $\int_0^w (Zh_0's(s) + h_0(s)Z')d\Lambda(s)$ is a Donsker and uniformly bounded. Therefore, $e^{t'Z\int_0^w (Zh_0'(s)+h_0(s)Z')d\Lambda(s)}$ is a Donsker.

$f(x) = x2$ is Lipschiz continuous on bounded sets, then $\{[1+(\beta - t)'h_0(w)]2 : \|\beta - \beta_0\| \le \eta, \|t - \beta_0\| < \epsilon\}$ is a Donsker, and then $\frac{h_0(w)h_0'(w)s}{[1+(\beta-t)'h_0(w)]2}$ is a Donsker.

Therefore,$\{\ddot{l}(t,\beta,\lambda), (t,\beta,\Lambda) \in V\}$ is a Donsker by Donsker preservation Thm and thus P_0-GC.

20.3.4 The likelihood for a sample of n iid observations $(U_1,\delta_1,Z_1), \cdots, (U_n,\delta_n,Z_n)$ is

$$l_n(\theta) = \mathbb{P}_n\{\delta(\log a(u) + \beta'Z) - (1+\delta)\log(1 + e^{\beta'Z}A(u))\}$$

and $l(\theta) = \delta(\log a(u) + \beta'Z) - (1+\delta)\log(1 + e^{\beta'Z}A(u))$,where a is the derivative of A.

$$
\begin{aligned}
U_1[\Psi](a) &= \frac{\partial}{\partial t} l(\theta + ta, A(\cdot))|_{t=0} \\
&= a'Z \left[\delta - (1+\delta) \frac{A(U)e^{\beta' Z}}{1 + e^{\beta' Z} A(U)} \right] \\
U_2[\Psi](h) &= \delta h(U) - (1+\delta) \frac{e^{\beta' Z} \int_0^u h(s) dA(s)}{1 + e^{\beta' Z} A(U)} \\
& \left(t \mapsto A_t(\cdot) = \int_0^{(\cdot)} (1 + th(s)) dA(s) \right) \\
U[\Psi](c) &= U_1[\Psi](a) + U_2[\Psi](h)
\end{aligned}
$$

Condition(20.2):$\{U[\Psi](c) : \|\Psi - \Psi_0\| \leq \epsilon, c \in C_p\}$ is a Donsker for some $\epsilon > 0$;

$Z \in \mathbb{R}^k$ in a compact set, then $a'Z$ is a Donsker; $f(x) = \frac{x}{1+x}$ is a monotone function, therefore $\frac{A(U)e^{\beta' Z}}{1+e^{\beta' Z} A(U)}$ is a Donsker and thus $U_1[\Psi](a)$ is a Donsker.

$h(s)$ has bounded variation, so it can be written as the difference of two monotone functions, and $h(s)$ is bounded Donsker. Similar argument as what we've done, we have $\left\{ \frac{1}{1+e^{\beta' Z} A(U)} \right\}$, $\{e^{\beta' Z}\}$, $\left\{ \int_0^u h(s) dA(s) \right\}$ are Donskers. Then $U_2[\Psi](h)$ is a Donsker.

Therefore, condition (20.2) follows.

Condition (20.3): $\sup_{c \in C_p} P_0 \left| U[\Psi](c) - U[\Psi_0](c) \right|^2 \to 0$ as $\Psi \to \Psi_0$;

$U[\Psi](c)$ is continuous, then $|U[\Psi](c) - U[\Psi_0](c)| \to 0$ as $\Psi \to \Psi_0$. Therefore, condition (20.3) follows.

Condition (20.4): $\sup_{c \in C_p} \|\sigma[\Psi](c) - \sigma[\Psi_0](c)\|_{(P)} \to 0$ as $\|\Psi - \Psi_0\|_{(P)} \to 0$. This follows because $U[\Psi](c)$ is continuous on Ψ.

By Thm 15.9, σ is continuously invertible and onto.By Thm 15.6,$\hat{\Psi}_n$ is uniformly consistent for Ψ_0. $P_n \Psi_n(\hat{\Psi}_n)(c) = 0 = o_p(n^{-\frac{1}{2}})$, because $\hat{\Psi}_n$ is the maximizer of the log-likelihood. Thus all conditions in Corollary 20.1 are satisfied, and $(\hat{\theta}_n, A_n)$ is asymptotically efficient.