Solutions to Problem Set 10

18.5.3 When S_n^2 is a consistent estimate of $\text{var}(\sqrt{n}T_n)$, $\frac{\sqrt{(n)}T_n}{S_n} > z_{1-\alpha}$ is asymptotically optimal at level α. Now,

$$\begin{aligned}
\sqrt{n}T_n &= \sqrt{n}\int_{\mathbb{R}} \text{sign}(x) d\mathbb{F}_n(x) \\
&= \sqrt{n}(P(x > 0) - P(x < 0)) \\
&= \sqrt{n}(1 - 2\mathbb{F}_n(0)) \\
&= \sqrt{n}(1 - 2 \cdot \frac{1}{n}\sum_{i=1}^{n} 1\{X_i \leq 0\});
\end{aligned}$$

$$\begin{aligned}
\text{var}(\sqrt{n}T_n) &= n \cdot \frac{4}{n2} \cdot n \cdot \text{var}(1\{x_i \leq 0\}) \\
&= 4P(x_i \leq 0)(1 - P(x_i \leq 0)) \\
&= 4F(0)(1 - F(0)).
\end{aligned}$$

$\Rightarrow S_n = 2\sqrt{F(0)(1 - F(0))}$.

18.5.4 ϕ is a map between normed spaces $\mathbb{D}$ and $l^\infty(\mathbb{D}_1')$:

$$\begin{aligned}
\phi(ad_1 + bd_2) &= \{d'(ad_1 + bd_2) : d' \in \mathbb{D}_1'\} \\
&= \{ad'd_1 + bd'd_2 : d' \in \mathbb{D}_1'\} \\
&= a\{d'd_1 : d' \in \mathbb{D}_1'\} + b\{d'd_2 : d' \in \mathbb{D}_1'\} \\
&= a\phi(d_1) + b\phi(d_2).
\end{aligned}$$

Therefore, ϕ is a linear operator. To show ϕ is continuous, we only need to show ϕ is bounded (Proposition 6.15). Accordingly,

$$\begin{aligned}
\|\phi\| &= \sup_{d\in\mathbb{D}:\|d\|\leq 1} \|\phi(d)\| \\
&= \sup_{d\in\mathbb{D}:\|d\|\leq 1} \sup_{d'\in\mathbb{D}_1':\|d'\|\leq 1} |d'(d)| \\
&\leq \sup_{d\in\mathbb{D}:\|d\|\leq 1} \|d\| \\
&\leq 1.
\end{aligned}$$

Thus, ϕ is continuous.

Since condition (18.6) holds, then

$$\|\phi(d)\| \;=\; \sup_{d' \in \mathbb{D}' : \|d'\| \leq 1} |d'(d)| \;\geq\; \frac{1}{c}\|d\|, \text{ where } 0 < c < \infty.$$

Thus ϕ has a continuous inverse ϕ^{-1} (Lemma 6.16).

For any $y_1, y_2 \in l^{\infty}(\mathbb{D}'_1)$, we can find $d_1, d_2 \in \mathbb{D}$ such that $\phi(d_1) = y_1$ and $\phi(d_2) = y_2$. Hence

$$\begin{aligned}
\phi^{-1}(ay_1 + by_2) &= \phi^{-1}(a\phi(d_1) + b\phi(d_2)) \\
&= \phi^{-1}(\phi(ad_1 + bd_2)) \\
&= ad_1 + bd_2 \\
&= a\phi^{-1}(y_1) + b\phi^{-1}(y_2).
\end{aligned}$$

Therefore, ϕ^{-1} is linear.