Solutions to Problem Set 9

15.6.2 Since $\left(1+e^{\beta' Z}A(U)\right)^{-1} = \left(1+e^{\beta' Z}A_0(U)\right)^{-1}$ almost surely, we then have $e^{\beta' Z}A(U) = e^{\beta_0' Z}A_0(U)$ almost surely. Because A_0 is continuous with density bounded above and below, we can get $e^{\beta' Z}dA(U) = e^{\beta_0' Z}dA_0(U)$. Thus, $e^{\beta' Z}\frac{dA(U)}{dA_0(U)} = e^{\beta_0' Z}$ almost surely. If we define $a(t) = \frac{dA(t)}{dA_0(t)}$, then this equation is equivalent to $e^{\beta' Z}a(t) = e^{\beta_0' Z}$ almost surely for every $t \in [0,\tau]$. Noctice that $A(t) = A_0(t)$ for all $t \in [0,\tau]$ is equivalent to $a(t) = 1$ for all $t \in [0,\tau]$. Therefore, if we can verify whenever $e^{\beta' Z}a(t) = e^{\beta_0' Z}$ almost surely for every $t \in [0,\tau]$, we have $\beta = \beta_0$, $a(t) = 1$, then the desired result follows. Accordingly, note

$$\begin{aligned}
& e^{\beta' Z}a(t) = e^{\beta_0' Z} \qquad (1)\\
\iff\ & \beta' Z + \log a(t) = \beta_0' Z\\
\iff\ & (\beta-\beta_0)' Z = -\log a(t)\\
\Rightarrow\ & \text{var}[(\beta-\beta_0)' Z] = \text{var}(-\log a(t)) = 0\\
\Rightarrow\ & (\beta-\beta_0)'\text{var}(Z)(\beta-\beta_0) = 0\\
\Rightarrow\ & \beta = \beta_0, \text{ since var}(Z)\text{is positive definite.}
\end{aligned}$$

Also, we can get $a(t) = 1$ from (1) if we set $\beta = \beta_0$.

17.4.4 For any $y \in L_2(0,1)$, $\int_0^1 y^2(u)du < \infty$. Consider $y_n(u) = y(u)I(\frac{1}{n} \le u < 1), n = 1,2,\cdots$ We will show:

(1) $y_n \in R(T)$

(2) $\|y_n - y\| \to 0$

and then $R(T)$ is dense in $L_2(0,1)$.

$$
\begin{aligned}
(1)\ \int_0^1 u^{-2}y_n^2(u)du &= \int_0^1 u^{-2}y^2(u)I(\frac{1}{n} \le u < 1)du \\
&= \int_{1/n}^1 u^{-2}y^2(u)du \\
&\le \left(\frac{1}{n}\right)^{-2} \int_{1/n}^1 y^2(u)du \\
&\le n^2 \int_0^1 y^2(u)du < \infty.
\end{aligned}
$$

Therefore, $y_n \in R(T)$. Also,

$$
\begin{aligned}
\|y_n - y\| &= \int_0^1 (y_n(u) - y(u))^2 du \\
&= \int_0^1 y^2(u)(I(\frac{1}{n} \le u < 1) - 1)^2 du \\
&= \int_0^1 y^2(u)I(0 \le u < \frac{1}{n})du \\
&= \int_0^{1/n} y^2(u)du \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Therefore, $R(T)$ is dense in $L_2(0,1)$.

17.4.5 Let $A : H_1 \to H_2$ and $A^* : H_2 \to H_1$. Decompose $D(A^*) \subset H_2$ (domain of A^*) into $R(A)$ and $R(A)^\perp$.

$\forall y \in D(A^*), y = y_1 + y_2$, where $y_1 \in R(A) \cap D(A^*)$ and $y_2 \in R(A)^\perp \cap D(A^*)$. Then we have:

$y_1 \in R(A) \Rightarrow \exists x \in A, \text{s.t.}\, y_1 = Ax,$
$y_2 \in R(A)^\perp = N(A^*) \Rightarrow A^*y_2 = 0.$

Since $A^*y = A^*(y_1 + y_2) = A^*y_1 + A^*y_2$, then $A^*y = A^*Ax \in R(A^*A)$, and thus $R(A^*) \subset R(A^*A)$.