Solutions to Problem Set 8

13.4.2 Assume $\Psi \in Z(\Theta, \mathbb{L})$, $\Psi_n, \Psi_n^\circ \in Z(\Theta, \mathbb{L})$ for all n large enough. Ψ_n° is a bottstrapped version of Ψ_n based on both the data sequence and a sequence of weights $W = \{W_n, n \geq 1\}$. Ψ is uniformly norm-bounded over Θ and it satisfies:

(1) $\Psi(\theta_0) = 0$;

(2) $\|\Psi(\theta_n)\|_{\mathbb{L}} \to 0$ implies $\|\theta_n - \theta_0\| \to 0$ for any $\theta_n \in \Theta$;

(3) Ψ is Fr*é*chet differentiable at θ_0 with continuously invertible derivative $\dot{\Psi}_{\theta_0}$.

Let $\tilde{\phi} \in \Phi(\Theta, \mathbb{L})$ and $\hat{\theta_n} = \tilde{\phi}(\Psi_n), \hat{\theta_n^\circ} = \tilde{\Phi}(\Psi_n^\circ)$. Assume that $r_n(\Psi_n - \Psi) \rightsquigarrow X$ in $l^\infty(\Theta, \mathbb{L})$ and $r_n c_0(\tilde{\Phi}(\Psi_n^\circ) - \Psi_n) \rightsquigarrow_W^p X$, with X tight and taking its values in $l^\infty(\Theta, \mathbb{L})$ for some sequence of constants $r_n \to \infty$ and $0 < c_0 < \infty$. The maps $W_n \longmapsto h(\hat{\Psi_n})$ are measurable for every $h \in C_0(l^\infty(\Theta, \mathbb{L}))$ out almost surely. Then:

$$r_n(\hat{\theta_n} - \theta_0) \rightsquigarrow \dot{\Psi}_{\theta_0}^{-1} X(\theta_0) \text{ and } r_n c_0(\hat{\theta_n^\circ} - \hat{\theta_n}) \rightsquigarrow_W^p \dot{\Psi}_{\theta_0}^{-1} X(\theta_0).$$

14.6.9 (a) Show $0 \geq \inf_{t\geq 0} A_n(t) \geq \inf_{t \geq 0} B_n(t)$.

Since $\hat{F_n}(t) \geq F_n(t)$, then $A_n(t) \geq B_n(t)$, and $\inf_{t\geq 0} A_n(t) \geq \inf_{t\geq 0} B_n(t)$.

Suppose $\inf_{t\geq 0} A_n(t) = \delta$, then $\hat{F_n}(t) - F(t) \geq \delta$. Here we choose any t_i, s.t $\hat{F_n}(t_i) = F_n(t_i)$.Thus, we have $F_n(t_i) - F(t_i) \geq \delta$. Letting $n \to \infty$, we obtain $\delta \leq 0$, but this is a contradiction.

Therefore, $0 \geq \inf_{t\geq 0} A_n(t) \geq \inf_{t\geq 0} B_n(t)$.

(b) i. If $\sup_{t\geq 0} A_n(t) = \delta < 0$, then $\hat{F_n}(t) - F_(t) \leq \delta$. Similarly as in (a), we choose any t_i, s.t. $\hat{F_n}(t_i) = F_n(t_i)$. Then $F_n(t_i) - F(t_i) \leq \delta$. Letting $n \to \infty$, we get $\delta \geq 0$, but this is a Contradiction. Therefore, $\sup_{t\geq 0} A_n(t) \geq 0$. Also, $\sup_{t\geq 0} B_n(t) \geq 0$.

ii. If $\sup_{t\geq 0} B_n(t) = 0$, then $F_n(t) - F(t) \geq -\epsilon$ for a small $\epsilon > 0$ and $F_n(t) - F(t) \leq 0$. Since $\hat{F_n}$ is the least concave majorant of $\mathbb{F}_n$, then we have $\hat{F}_n(t) - F(t) \geq F_n(t) - F(t) \geq -\epsilon$. $F(t)$ is a concave function because f is differentiable at t with derivative $-\infty < f'(t) < 0$. $F(t) \geq F_n(t)$ indicates that $\hat{F}_n(t) \leq F(t)$, i.e., $\hat{F}_n(t) - F(t) \leq 0$. Thus $\sup t \geq 0 A_n(t) \leq 0$. From i, we have $\sup t \geq 0 A_n(t) \geq 0$, therefore $\sup t \geq 0 A_n(t) = 0$.

iii. If $\sup_{t\geq 0} B_n(t)$, suppose that $\sup_{t\geq 0}(F_n(t) - F(t)) = \delta > 0$, then $F_n(t) - F(t) \leq \delta$. Thus $F(t)$ is a concave function, and $F(t) \geq F_n(t) - \delta$. Since $\hat{F}_n(t)$ is the least concave majorant of $F_n(t)$, we have $\hat{F}_n(t) - \delta$ is the least concave majorant of $F_n(t) - \delta$.

Therefore, $\hat{F}_n(t) - \delta \leq F(t)$, i.e., $\hat{F}_n(t) - F(t) \leq \delta = \sup_{t\geq 0} B_n(t)$. $\sup_{t\geq 0} A_n(t) \leq \sup_{t\geq 0} B_n(t)$ follows.

From i, ii, and iii, we can obtain $0 \leq \sup_{t\geq 0} A_n(t) \leq \sup_{t\geq 0} B_n(t)$.

(c) Now we prove $\sup_{t\geq 0} |A_n(t)| \leq \sup_{t\geq 0} |B_n(t)|$. Notice that $A_n(t) \geq B_n(t)$.

If $A_n(t) \geq B_n(t) \geq 0$, then the result follows from (b).

If $A_n(t) \geq 0 \geq B_n(t)$, then $\hat{F}_n(t) \geq F(t)$ and $F_n(t) \leq F(t)$. However, $\hat{F}_n(t) \leq F(t)$. This is because $F(t) \geq F_n(t)$ and F(t) is a concave function. This is a contradiction.

If $0 \geq A_n(t) \geq B_n(t)$, then $\sup_{t\geq 0} |A_n(t)| = \sup_{t\geq 0}(-A_n(t)) = -\inf_{t\geq 0} A_n(t) \leq -\inf_{t\geq 0} B_n(t) = \sup_{t\geq 0} |B_n(t)|$.

14.6.10 First, we will prove the process $h \mapsto \mathbb{Z}(\sigma h - \mu)$ is equal in distribution to the process $h \mapsto \sqrt{\sigma}\mathbb{Z}(h) - \mathbb{Z}(\mu)$.

Since $\{\mathbb{Z}(h) : h \in \mathbb{R}\}$ is zero-mean Gaussian and the increment $\mathbb{Z}(g) - \mathbb{Z}(h)$ has variance $|g - h|$, then we have:

$$\mathrm{var}(\mathbb{Z}(\sigma h - \mu)) = \mathrm{var}(\mathbb{Z}(\sigma h - \mu) - \mathbb{Z}(0)) = |\sigma h - \mu|$$

$$\mathrm{var}(\mathbb{Z}(\sigma h) - \mathbb{Z}(\mu)) = |\sigma h - \mu|$$
$$E(\mathbb{Z}(\sigma h - \mu)) = E(\mathbb{Z}(\sigma h) - \mathbb{Z}(\mu)) = 0,$$

therefore, $\mathbb{Z}(\sigma h - \mu) =^d \mathbb{Z}(\sigma h) - \mathbb{Z}(\mu)$.

We also have:

$$\mathrm{var}(\mathbb{Z}(\sigma h)) = \mathrm{var}(\mathbb{Z}(\sigma h) - \mathbb{Z}(0)) = |\sigma h| \text{ and}$$
$$\mathrm{var}(\sqrt{\sigma}\mathbb{Z}(\sigma h)) = |\sigma|\mathrm{var}(\mathbb{Z}(h)) = |\sigma h|.$$

Therefore, $\mathbb{Z}(\sigma h) =^d \sqrt{\sigma}\mathbb{Z}(h) - \mathbb{Z}(\mu)$, and $\mathbb{Z}(\sigma h - \mu) =^d \sqrt{\sigma}\mathbb{Z}(h) - \mathbb{Z}(\mu)$ follows.

Define $h = \left(\frac{a}{b}\right)^{\frac{2}{3}} g - \frac{c}{2b}$, then we have:

$$\begin{aligned}
& \mathrm{argmax}_h \{a\mathbb{Z}(h) - bh2 - ch\} \\
= \;& \mathrm{argmax}_h \left\{ a\mathbb{Z}\left(\left(\frac{a}{b}\right)^{\frac{2}{3}} g - \frac{c}{2b}\right) - b\left(\left(\frac{a}{b}\right)^{\frac{2}{3}} g - \frac{c}{2b}\right) 2 - c\left(\left(\frac{a}{b}\right)^{\frac{2}{3}} g - \frac{c}{2b}\right)\right\} \\
= \;& \left(\frac{a}{b}\right)^{\frac{2}{3}} \mathrm{argmax}_g \left\{ a\mathbb{Z}\left(\left(\frac{a}{b}\right)^{\frac{2}{3}} g - \frac{c}{2b}\right) - b\left(\left(\frac{a}{b}\right)^{\frac{2}{3}} g - \frac{c}{2b}\right)\left(\left(\frac{a}{b}\right)^{\frac{2}{3}} g - \frac{c}{2b}\right)\right\} - \frac{c}{2b} \\
= \;& \left(\frac{a}{b}\right)^{\frac{2}{3}} \mathrm{argmax}_g \left\{ a\mathbb{Z}\left(\left(\frac{a}{b}\right)^{\frac{2}{3}} g - \frac{c}{2b}\right) - b\left(\frac{a}{b}\right)^{\frac{4}{3}} g2\right\} - \frac{c}{2b} \\
=^d \;& \left(\frac{a}{b}\right)^{\frac{2}{3}} \mathrm{argmax}_g \left\{ a\left(\left(\frac{a}{b}\right)^{\frac{1}{3}} \mathbb{Z}(g) - \mathbb{Z}\left(\frac{c}{2b}\right)\right) - b\left(\frac{a}{b}\right)^{\frac{4}{3}} g2\right\} - \frac{c}{2b} \\
= \;& \left(\frac{a}{b}\right)^{\frac{2}{3}} \mathrm{argmax}_g \{\mathbb{Z}(g) - g2\} - \frac{c}{2b}.
\end{aligned}$$