Solutions to Problem Set 7

10.5.1 (a) Recall that $\|\xi\|_{2,1} = \int_0^\infty \sqrt{P(|\xi| > u)}du$. However, this is not norm, because it does not satisfy the triangle inequality.

(b)For the first inequality, if we can prove $E(\xi^2) = 2\int_0^\infty P(|\xi| > u)udu \leq 2\|\xi\|_{2,1} \times \|\xi\|_2$, then we have $\|\xi\|_2^2 = E(\xi^2) \leq 2\|\xi\|_{2,1} \times \|\xi\|_2$, which gives $\frac{1}{2}\|\xi\|_2 \leq \|\xi\|_{2,1}$. Hence, what is left is to verify these inequalities.

i) First:

$$\begin{aligned} 2\int_0^\infty P(|\xi| > u)udu &= 2\int_0^\infty u \int_u^\infty f(\xi)\,d\xi du + 2\int_0^\infty u \int_{-\infty}^{-u} f(\xi)\,d\xi du \\ &= 2\int_0^\infty f(\xi)\int_0^\xi u\,dud\xi + 2\int_0^\infty f(\xi)\int_{-\xi}^0 u\,dud\xi \\ &= \int_{-\infty}^\infty \xi^2 f(\xi)d\xi \\ &= E(\xi^2) \end{aligned}$$

ii) Second:

$$\begin{aligned} 2\int_0^\infty P(|\xi| > u)udu &= 2\int_0^\infty \sqrt{P(|\xi| > u)}\sqrt{P(|\xi| > u)}udu \\ &\leq 2\int_0^\infty \sqrt{P(|\xi| > u)}\sqrt{\frac{E|\xi|^2}{u^2}}udu \\ &= 2\int_0^\infty \sqrt{P(|\xi| > u)}\left(E|\xi|^2\right)^{\frac{1}{2}}du \\ &= 2\|\xi\|_{2,1} \times \|\xi\|_2 \end{aligned}$$

Here, the inequality follows from Markov's inequality.

For the second inequality,for any $a > 0$:

$$\begin{aligned}
\|\xi\|_{2,1} &= \int_0^a \sqrt{P(|\xi| \geq u)}du + \int_a^\infty \sqrt{P(|\xi| \geq u)}du \\
&\leq a + \int_a^\infty \left(\frac{\|\xi\|_r^r}{u^r}\right)^{\frac{1}{2}} du \\
&= a + \|\xi\|_r^{\frac{r}{2}} \cdot \int_a^\infty u^{-\frac{r}{2}} du \\
&= a + \|\xi\|_r^{\frac{r}{2}} \cdot \frac{2}{r-2} \cdot a^{1-\frac{r}{2}}
\end{aligned}$$

If we let $a = \|\xi\|_r$, then we have $\|\xi\|_{2,1} \leq \|\xi\|_r + \|\xi\|_r^{\frac{r}{2}} \cdot \frac{2}{r-2} \cdot \|\xi\|^{1-\frac{r}{2}} = \frac{r}{r-2}\|\xi\|_r$.

12.3.2 (a)We have

$$(\phi(B) - \phi(A))(s,t] = \frac{\phi(B)(t)}{\phi(B)(s)} - \frac{\phi(A)(t)}{\phi(A)(s)};$$

also, from the Duhamel equation, we have:

$$\begin{aligned}
(\phi(B) - \phi(A))(s,t] &= \int_{(s,t]} \phi(A)(0,u)\phi(B)(u,t](B-A)(du) \\
&= \int_{(s,t]} \frac{\phi(A)(u-)}{\phi(A)(0)} \cdot \frac{\phi(B)(t)}{\phi(B)(u)}(B-A)(du)
\end{aligned}$$

Let $S = 0$, and $\phi(A)(0) = 1$ for any $A \subset D(0,b]$. Then:

$$\phi(B)(t) - \phi(A)(t) = \int_{(0,t]} \phi(A)(u-)\frac{\phi(B)(t)}{\phi(B)(u)}(B-A)(du).$$

If $B = 0$, then $\phi(B)(t) = \exp(B^c(t)) \prod_{0<s\leq t}(1+\Delta B(s)) = \exp(B^c(t)) \prod_{0<s\leq t}(1+0) = 1$.Thus, we can get:

$$\phi(A)(t) = 1 + \int_{(0,t]} \phi(A)(u-)Adu.$$

Therefore,$\phi(A)(s,t] = 1 + \int_{(s,t]} \phi(A)(s,u)Adu$, and this finishes the proof.

(b) From the uniqueness of (a),if we can prove

$$\phi(A)(s,t] = 1 + \sum_{m=1}^{\infty} \int_{s<t_1<\cdots<t_m<t} A(dt_1)\cdots A(dt_m)$$

satisfy the equation $B(s,t] = 1 + \int_{(s,t]} B(s,u)A(du)$, then the result follows.

$$\begin{aligned}
\text{RHS} &= 1 + \int_{(s,t]} \left(1 + \sum_{m=1}^{\infty} \int_{s<t_1<\cdots<t_m<u} A(dt_1)\cdots A(dt_m)\right) Adu \\
&= 1 + \int_{(s,t]} Ad(u) + \int_{(s,t]} \left(\sum_{m=1}^{\infty} \int_{s<t_1<\cdots<t_m<u} A(dt_1)\cdots A(dt_m)\right) Adu \\
&= 1 + \int_{(s,t]} Ad(u) + \int_{(s,t]} \left(\sum_{m=1}^{\infty} \int_{s<t_1<\cdots<t_m<t_{m+1}} A(dt_1)\cdots A(dt_m)\right) A(dt_{m+1}) \\
&= 1 + \int_{(s,t]} Ad(u) + \sum_{m=1}^{\infty} \int_{s<t_1<\cdots<t_m<t_{m+1}<t} A(dt_1)\cdots A(dt_m)A(dt_{m+1}) \\
&= \phi(A)(s,t] \\
&= \text{LHS}
\end{aligned}$$

Here, the fourth equation follows from Fubini's theorem. Therefore, $\phi(A)(s,t]$ is equivalent to Peano series representation.

(c) It will be suffice to show

$$\phi(A)(s,t] = 1 + \sum_{m=1}^{\infty} \int_{s<t_1<\cdots<t_m<t} A(dt_1)\cdots A(dt_m)$$

satisfies this "backward" Volterra integral equation, and the proof can be done similarly as part (b).

Another approach:
Denote a new process $C(0,u] = B(t-u,t]$, then from part (a), $\phi(A)(0,s-t]$ is equivalent to the unique solution C of the following Volterra integral equation:

$$C(0,t-s] = 1 + \int_{(0,t-s]} C(0,v)A(dv).$$

Then it is also the unique solution of equation:

$$B(s,t] = 1 + \int_{(0,t-s]} B(t-v,t]A(dv).$$

Let $u = -v$, then

$$B(s,t] = 1 - \int_{(t,s]} B(u,t]A(du) = 1 + \int_{(s,t]} B(u,t]A(du).$$

Since $\phi(A)(0, s-t]$ and $\phi(A)(s,t]$ are equivalent, then the desired result follows.