Solutions to Problem Set 6

9.6.6 $\mathcal{C}_1$ can pick out $O(n^{V_1-1})$ subsets from $\mathcal{X}_1$, then $\mathcal{C}_1 \bigcup \mathcal{X}_2$ can pick out $O(n^{V_1-1})$ from $\mathcal{X}_1 \bigcup \mathcal{X}_2$, similarly, $\mathcal{C}_2 \bigcup \mathcal{X}_1$ can pick out $O(n^{V_2-1})$ subsets from $\mathcal{X}_1 \bigcup \mathcal{X}_2$
$\Rightarrow (\mathcal{C}_1 \bigcup \mathcal{X}_2) \bigcap (\mathcal{C}_2 \bigcup \mathcal{X}_1) = (\mathcal{C}_1 \bigcap \mathcal{C}_2) \bigcup (\mathcal{X}_2 \bigcap \mathcal{X}_1)$ can pick out $O(n^{V_1+V_2-1})$ subsets from $\mathcal{X}_1 \bigcup \mathcal{X}_2$;

$$\begin{aligned}
\left[\left(\mathcal{C}_1 \bigcup \mathcal{C}_2\right) \bigcap \left(\mathcal{X}_2 \bigcup \mathcal{X}_1\right)\right]^c &= \left(\mathcal{C}_1 \bigcup \mathcal{C}_2\right)^c \bigcup \left(\mathcal{X}_2 \bigcup \mathcal{X}_1\right)^c \\
&= \left(\mathcal{C}_1^c \bigcap \mathcal{C}_2^c\right) \bigcup \left(\mathcal{X}_1^C \bigcap \mathcal{X}_2^c\right) \\
&= \left(\mathcal{C}_1^c \bigcap \mathcal{C}_2^c\right) \bigcup \left(\mathcal{X}_2 \bigcap \mathcal{X}_1\right) \\
&= \left(\mathcal{C}_1^c \bigcup \mathcal{X}_2\right) \bigcap \left(\mathcal{C}_2^c \bigcup \mathcal{X}_1\right) \\
&= \left(\mathcal{C}_1^c \bigcap \mathcal{C}_2^c\right) \bigcup \left(\mathcal{X}_2 \bigcap \mathcal{X}_1\right)
\end{aligned}$$

$V(\mathcal{C}_1^c) = V(\mathcal{C}_1), V(\mathcal{C}_2^c) = V(\mathcal{C}_2)$,
$\Rightarrow [(\mathcal{C}_1 \bigcup \mathcal{C}_2) \bigcap (\mathcal{X}_2 \bigcup \mathcal{X}_1)]^c$ can pick out $O(n^{V_1+V_2-1})$ subsets from $\mathcal{X}_1 \bigcup \mathcal{X}_2$,
and since $V\left[(\mathcal{C}_1 \bigcup \mathcal{C}_2) \bigcap (\mathcal{X}_2 \bigcup \mathcal{X}_1)\right] = V\left[(\mathcal{C}_1 \bigcup \mathcal{C}_2) \bigcap (\mathcal{X}_2 \bigcup \mathcal{X}_1)\right]^c$
$\Rightarrow (\mathcal{C}_1 \bigcup \mathcal{C}_2) \bigcap (\mathcal{X}_2 \bigcup \mathcal{X}_1)$ can pick out $O(n^{V_1+V_2-1})$ subsets from $\mathcal{X}_1 \bigcup \mathcal{X}_2$.
$\Rightarrow \mathcal{C}_1 \bigsqcup \mathcal{C}_2$ is VC on $\mathcal{X}_1 \bigcup \mathcal{X}_2$ with VC index $V_1 + V_2 - 1$.
$\Rightarrow$ Similarly, we can get $\mathcal{C}_1 \bigsqcup \mathcal{C}_2 \bigsqcup \mathcal{C}_3$ is VC on $\mathcal{X}_1 \bigcup \mathcal{X}_2 \bigcup \mathcal{X}_3$ with VC index $V_1 + V_2 + V_3 - 2$, as this goes on, finally,
$\Rightarrow \bigsqcup_{i=1}^m \mathcal{C}_i$ is a VC in $\mathcal{X}_1 \bigcup \mathcal{X}_2$ with VC index $V_1 + V_2 + \cdots + V_m - m + 1$.

9.6.10 Fix $\epsilon > 0$ let n be the smallest integer $\geq \frac{3}{\epsilon}$. For any $p = (k_0, \cdots, k_n) \in P \equiv \{1, \cdots, n\}^{n+1}$,define the path $\bar{p}$ to be the collection of all function in $\mathcal{F}$ such that $f \in \bar{p}$ only if $f(\frac{i}{n}) \in [\frac{k_i - 1}{n}, \frac{k_i}{n}]$ for all i=0$\cdots n$. Since $\|f(x) - f(y)\| \leq |x - y|$, then if $f(\frac{i}{n}) \in [\frac{j}{n}, \frac{j+1}{n}]$, that is, $\frac{j}{n} \leq f(\frac{i}{n}) \leq \frac{j+1}{n}$ $\Rightarrow$

$$f\left(\tfrac{i}{n}\right) - \left(\tfrac{i+1}{n} - \tfrac{i}{n}\right) \leq f\left(\tfrac{i+1}{n}\right) \leq f\left(\tfrac{i}{n}\right) + \left(\tfrac{i+1}{n} - \tfrac{i}{n}\right) \text{i.ee} \tfrac{j-1}{n} \leq f\left(\tfrac{i+1}{n}\right) \leq \tfrac{j+2}{n}$$

Since $f : [0, 1] \to [0, 1]$, then

$$f\left[\tfrac{i+1}{n}\right] \in \left[\tfrac{(j-1)\bigvee 0}{n}, \tfrac{(j+2)\bigwedge n}{n}\right].$$

Therefore,to include all elements of $\mathcal{F}$, if $f(\frac{0}{n}) \in [0,1]$, then $f(\frac{0}{n}) \in [\frac{k_0-1}{n}, \frac{k_0}{n}] \subset [0,1]$, k_0 has n choices; since $f(\frac{1}{n}) \in [\frac{k_0-2}{n}, \frac{k_0+1}{n}]$ and $f(\frac{1}{n}) \in [\frac{k_1-1}{n}, k_1 n]$, then there exist at most three choices of k_1, and then similarly, we have at most three choices on k_2 if given k_0 and k_1.
$\Rightarrow$ At most $n \cdot 3^n$ paths are needed to include all elements of $\mathcal{F}$.
For each path $\bar{p}$, $p = (k_0, \cdots, k_n)$, if $x \in [\frac{i}{n}, \frac{i+1}{n}]$we define:

$$\begin{aligned} L_p(x) &= \frac{k_i - 2}{n} \bigvee 0 \\ U_p(x) &= \frac{k_i + 1}{n} \bigwedge 1 \end{aligned}$$

$L_p(x), U_p(x)$ are right continuous, and $0 \leq L_p(x) \leq U_p(x) \leq 1, U_p(x) = L_p(x) \leq \frac{3}{n} \leq \epsilon$.
Actually:
if $x \in [\frac{i}{n}, \frac{i+1}{n}]$,$f(\frac{1}{n}) \in [\frac{k_0-2}{n}, \frac{k_0+1}{n}]$, $f(\frac{i}{n}) \in [\frac{k_i-1}{n}, \frac{k_i}{n}] \subset [L_p(x), U_p(x)]$, then

$$|f(x) - f(\tfrac{i}{n})| \leq |x - \tfrac{1}{n}| \leq \tfrac{1}{n} \Rightarrow f(x) \in [\tfrac{k_i-2}{n}, \tfrac{k_i+1}{n}] \subset [L_p(x), U_p(x)]$$

since $n \leq \frac{3}{\epsilon}$, then

$$\begin{aligned} n \cdot 3^n &< \left(1 + \frac{3}{\epsilon}\right) \exp\left\{(1 + \frac{3}{\epsilon}\log 3)\right\} \\ &= \exp\left\{\log\left(1 + \frac{3}{\epsilon}\right) + \left(1 + \frac{3}{\epsilon}\log 3\right)\right\} \\ &< \exp\left\{\frac{3}{\epsilon} + \log 3 + 3\log 3 \cdot \frac{1}{\epsilon}\right\} \\ &< \exp\left(c \cdot \frac{1}{\epsilon}\right) \text{ for some } c \end{aligned}$$