Solutions to Problem Set 5

8.5.1 (a) i) Clearly, $\|X\|_\psi = 0$, then $\psi\left(\frac{\|X\|}{C}\right) = 0$, i.e., $X = 0$ a.s.

ii)

$$\begin{aligned}
\|aX\|_\psi &= \inf\left\{c > 0 : \mathbf{E}\psi\left(\frac{\|aX\|}{c}\right) \leq 1\right\} \\
&= \inf\left\{c > 0 : \mathbf{E}\psi\left(\frac{\|X\|}{c/|a|}\right) \leq 1\right\} \\
&= |a| \inf\left\{\frac{c}{|a|} > 0 : \mathbf{E}\psi\left(\frac{\|X\|}{c/|a|}\right) \leq 1\right\} \\
&= |a|\|X\|_\psi.
\end{aligned}$$

iii) Since ψ is a convex function, then if $\forall X_1, X_2 \in H_\psi, c_1, c_2 > 0$ satisfy $\mathbf{E}\psi\left(\frac{\|c_1 X\|}{c_1}\right) \leq 1, \mathbf{E}\psi\left(\frac{\|c_2 X\|}{c_2}\right) \leq 1$, then for $0 \leq a_1 \leq 1$,

$$\begin{aligned}
&\mathbf{E}\psi\left(\frac{|X_1|}{c_1} \cdot a_1 + \frac{|X_2|}{c_2} \cdot (1 - a_1)\right) \\
&\qquad \leq \mathbf{E}\left[a_1\psi\left(\frac{|X_1|}{c_1}\right) + (1 - a_1)\psi\left(\frac{|X_2|}{c_2}\right)\right] \\
&\qquad = a_1\mathbf{E}\left(\psi\left(\frac{|X_1|}{c_1}\right)\right) + (1 - a_1)\mathbf{E}\left(\psi\left(\frac{|X_2|}{c_2}\right)\right) \\
&\qquad \leq 1,
\end{aligned}$$

therefore, $\left\{X : \mathbf{E}\psi\left(\frac{|X|}{c}\right) \leq 1\right\}$ is a convex set.

$\forall X_1, X_2 \in H_\psi, \text{and} \forall \epsilon > 0, \mathbf{E}\psi\left(\frac{X_1}{\|X_1\|_\psi + \epsilon}\right) \leq 1, \mathbf{E}\psi\left(\frac{X_2}{\|X_2\|_\psi + \epsilon}\right) \leq 1,$
$\Rightarrow$
$\mathbf{E}\left(\psi\left(\frac{\|X_1\|_\psi + \epsilon}{\|X_1\|_\psi + \|X_2\|_\psi + 2\epsilon} \cdot \frac{X_1}{\|X_1\|_\psi + \epsilon}\right) + \psi\left(\frac{\|X_2\|_\psi + \epsilon}{\|X_1\|_\psi + \|X_2\|_\psi + 2\epsilon} \cdot \frac{X_2}{\|X_2\|_\psi + \epsilon}\right)\right)$
≤ 1, then we can get

$$\mathbf{E}\left(\psi\left(\frac{X_1 + X_2}{\|X_1\|_\psi + \|X_2\|_\psi + 2\epsilon}\right)\right) \leq 1$$

$\Rightarrow \|X_1+X_2\|_\psi \le \|X_1\|_\psi+\|X_2\|_\psi+2\epsilon \Rightarrow \|X_1+X_2\|_\psi \le \|X_1\|_\psi+\|X_2\|_\psi$.

(b) For any Cauchy sequence of random variables $\{X_n\} \in H_\psi$,$\|X_m - X_n\|_\psi \to 0$, $X_n s$ are random variables, then they are asymptotically measurable.
$\forall \epsilon > 0, \exists M$, when $m, n \ge M$, $\|X_m - X_n\|_\psi < \epsilon$,therefore:

$$\|X_n\|_\psi \le \|X_n - X_M\|_\psi + \|X_M\|_\psi < \|X_M\|_\psi + \epsilon < \infty,$$

that is: $\limsup_{n\to\infty} \|X_n\|_\psi < \infty$, then we can set $\|X_n\|_\psi \le M$.
$\forall\, \epsilon > 0$, find c_0, s.t $\psi\left(\frac{c_0}{M} > \frac{1}{\epsilon}\right)$, since ψ is nondecreasing,then

$$\begin{aligned} P\left(|X_n| \le c_0\right) &= P\left(\frac{|X_n|}{M} \le \frac{c_0}{M}\right) \\ &= P\left(\psi\left(\frac{|X_n|}{M}\right) \le \psi\left(\frac{c_0}{M}\right)\right) \\ &= 1 - P\left(\psi\left(\frac{|X_n|}{M}\right) > \psi\left(\frac{c_0}{M}\right)\right) \\ &\ge 1 - \frac{\mathbf{E}\psi(\frac{|X_n|}{M})}{\psi(\frac{c_0}{M})} \\ &= 1 - \epsilon. \end{aligned}$$

Then X_n is asymptotically tight.
From Prohorov's theorem, it has a subsequence $X_n' \rightsquigarrow$ a tight Borel law, i.e., $X_n' - X \rightsquigarrow 0$, then we can get $X_n' - X \to^P 0$ so there exists a subsequence $\{X_n''\} \subset X_n'$, and $X_n'' - X \to^{a.s.} 0$, and then $\|X\|_\psi \le \|X_n'' - X\|_\psi + \|X_n''\|_\psi < \infty$, which means $X \in H_\psi$.
If $X_n \to X, X_n \to Y$, then

$$\|X - Y\|_\psi \le \|X - X_n\|_\psi + \|Y - X_n\|_\psi \to 0$$

$\Rightarrow X = Y$. Thus X is unique, and the result follows.

8.5.7 The Orlicz norm $\|\cdot\|_{\psi_2(x)}$ is taken over $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ with $X_1, \cdots, X_n$ still fixed, if $P(|\sum_{i=1}^n \epsilon_i a_i| > x) \le 2e^{-x2/(2\|a\|^2)}$. Compared with Lemma

8.1, we can get $k = 2, p = 2, c = \frac{1}{2\|a\|^2}$,
$\Rightarrow$

$$\|X\|_{\psi_2} \leq \left(\frac{1+2}{\frac{1}{2\|a\|^2}} \right)^{\frac{1}{2}} = \sqrt{6}\|a\| \text{ and } \left\| \sum_{i=1}^{n} \epsilon_i a_i \right\|_{\psi_2} \leq \sqrt{6}\|a\|.$$

By lemma 8.7,

$$\begin{aligned}
\left\| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(x_i) \right\|_{\psi_2|X} &= \left\| \sum_{i=1}^{n} \epsilon_i \frac{f(x_i)}{n} \right\|_{\psi_2|X} \\
&\leq \sqrt{6} \left\| \frac{f(X)}{n} \right\| \\
&= \sqrt{6} \left(\sum_{i=1}^{n} \frac{f2(X_i)}{n2} \right)^{\frac{1}{2}} \\
&= \sqrt{\frac{6}{n}} \left(\frac{1}{n} \sum_{i=1}^{n} f2(X_i) \right)^{\frac{1}{2}} \\
&= \sqrt{\frac{6}{n}} \left(\mathbb{P}_n f2 \right)^{\frac{1}{2}}.
\end{aligned}$$