Solutions to Problem Set 4

7.5.5 We will apply the extended continuous mapping theorem with the additional assumption that T_0 is totally bounded by ρ (which should have been a condition in the statement of the problem). It suffices to show that $\sup_{t\in T_n} x_n(t) \to \sup_{t\in T_0} x(t)$ for any $x_n \to x$, where $x_n \to x, \{x_n\} \in l^\infty(T)$ and $x \in UC(T,\rho)$.

Since $T_n \to T_0$, we have for any $t \in T_0$ that $\exists\{t_n\} \in T_n$ such that $\rho(t_n, t) \to 0$. Thus for $x \in UC(T,\rho)$, we have for the sequence $\{t_n\}$, that $x(t_n) \to x(t)$. Moreover, we have for any $x_n \to x$, that

$$|x_n(t_n) - x(t)| \le |x_n(t_n) - x(t_n)| + |x(t_n) - x(t)| \to 0,$$

which means $x_n(t_n) - \epsilon < x(t) < x_n(t_n) + \epsilon, \epsilon > 0$ when n large enough. Then $x(t) \le \lim_{n\to\infty} \sup_{t\in T_n} x_n(t)$ since for each $t \in T_0$, we can find such a sequence t_n for which $x(t) \le \lim_{n\to\infty} \sup_{t\in T_n} x_n(t)$. Thus

$$\sup_{t\in T_0} x(t) \le \lim_{n\to\infty} \sup_{t\in T_n} x_n(t).$$

For each T_n, we have $\forall\epsilon > 0, \exists\{t_n\} \in T_n$:

$$\sup_{t_n\in T_n} x_n(t_n) - \epsilon \le x_n(t_n) \le \sup_{t_n\in T_n} x_n(t_n),$$

for all n large enough. By the total boundedness of T_0, such a sequence t_n will have a further subsequence that converges to a $t_0 \in T_0$. Suppose instead that $t_n \to t_1 \in A$, where $A \subset T - T_0$ and A is closed (since we can always take its closure). Then, since $T_n \to T_0$, $A \cap T_n = \emptyset$ for all n large enough, which leads to a contradiction. Thus $t_0 \in T_0$.

Hence, $\sup_{t_n\in T_n} x_n(t_n) - \epsilon \le x_n(t_0) \le \sup_{t_n\in T_n} x_n(t_n)$, $\forall\epsilon > 0, x_n(t_0) < x(t_0) + \epsilon$ for n large enough, and thus $\sup_{t_n\in T_n} x_n(t_n) - \epsilon \le x_n(t_0) < x(t_0) + \epsilon$, i.e $\sup_{t_n\in T_n} x_n(t_n) < x(t_0) + 2\epsilon$, which implies

$$\sup_{t\in T_0} x(t) \ge \lim_{n\to\infty} \sup_{t\in T_n} x_n(t).$$

Thus $\lim_{n\to\infty} \sup_{t\in T_n} x_n(t) = \sup_{t\in T_0} x(t)$, and the desired conclusions follow.

7.5.7 i)

$$\begin{aligned}
|X_n - X|^* &= (X_n - X)^* \vee [-(X_n - X)]* \\
&= (X_n - X)^* \vee [-(X_n - X)_*] \\
&= (X_n + (-X))^* \vee [-(X_n + (-X))_*] \\
&= (X_n^* - X) \vee [-(X_{n*} - X)] \\
&= |X_n^* - X| \vee |X_{n*} - X|.
\end{aligned}$$

Suppose $X_n \to^{as*} X$. Then there exists a sequence $\triangle_n$ of measurable random variables with $d(X_n, X) = |X_n - X| \leq \triangle_n$ for all n and with $P(\limsup_{n\to\infty} \triangle_n = 0 = 1$. Since $|X_n - X| \leq \triangle_n \Rightarrow |X_n - X|^* \leq \triangle_n$,

$$\begin{aligned}
\limsup_{n\to\infty} \triangle_n = 0 &\Rightarrow \limsup_{n\to\infty} |X_n - X|^* = 0 \\
&\Leftrightarrow \limsup_{n\to\infty} |X_n^* - X| \vee |X_n^* - X| = 0 \\
&\Leftrightarrow \limsup_{n\to\infty} |X_n^* - X| = 0 \,\text{and}\, \limsup_{n\to\infty} |X_n^* - X| = 0 \\
\Rightarrow P(\lim_{n\to\infty} |X_n^* - X| = 0) &\geq P(\limsup_{n\to\infty} |X_n^* - X| = 0) \\
&\geq P(\limsup_{n\to\infty} \triangle_0 = 0) = 1.
\end{aligned}$$

$\Rightarrow X_n^* \to^{as} X$, and similarly, $X_{n*} \to^{as} X$.

If $X_n^* \to^{as} X, X_{n*} \to^{as} X$, then
$\exists A \subset \Omega, P(A) = 1, \limsup_{n\to\infty} |X_n^* - X| = 0$ and
$\exists B \subset \Omega, P(B) = 1, \limsup_{n\to\infty} |X_{n*} - X| = 0$.
Then $\limsup_{n\to\infty} |X_n^* - X| = 0, \limsup_{n\to\infty} |X_{n*} - X| = 0$ on $A \cap B \subset \Omega, P(A \cap B) = 1$, so $\limsup_{n\to\infty} |X_n - X|^* = 0, X_n \to^{as*} X$ for $\triangle_n = |X_n - X|^*$.

ii) If $X_n \rightsquigarrow X$, then from the extended almost sure representation theorem there exists a new probability space $(\widetilde{\Omega}, \widetilde{A}, \widetilde{P})$ and a perfect sequence of maps ϕ_n such that $\widetilde{X_n} = X_n \circ \phi_n \to^{as*} \widetilde{X_n}$.

From i), we know $\widetilde{X_n} \to^{as*} \widetilde{X} \Rightarrow \widetilde{X_n}^* \to^{as} \widetilde{X} \Rightarrow \widetilde{X_n}^* \rightsquigarrow \widetilde{X}$. Since ϕ_n is perfect and $\widetilde{X_n}^* = X_n^* \circ \phi_n$, we have

$$Ef(\widetilde{X_n}^*) = \int f \circ X_n * \circ \phi_n d\widetilde{P} = \int f \circ X_n d\widetilde{P} \circ \phi_n^{-1} = Ef(X_n^*).$$

We can find a perfect map ϕ such that $\widetilde{X} = X \circ \phi$ and $Ef(\widetilde{X}) = Ef(X)$. Thus $Ef(X_n^*) = Ef(\widetilde{X_n}^*) \to Ef(\widetilde{X}) = Ef(X)$ which implies $X_n^* \rightsquigarrow X$.

Since $\widetilde{X}_{n*} = X_{n*} \circ \phi$, then similarly we know $X_{n*} \rightsquigarrow X$.
If $X_n^* \rightsquigarrow X$ and $X_{n*} \rightsquigarrow X$, we have

$$P(X_n^* \leq x) \leq P^*(X_n \leq x) \leq P(X_{n*} \leq x)$$

$\Rightarrow \lim_{n\to\infty} P^*(X_n \leq x) = F_X(x) = P(X \leq x)$. Since also

$$P(X_n^* \leq x) \leq P_*(X_n \leq x) \leq P(X_{n*} \leq x),$$

$\Rightarrow \lim_{n\to\infty} P_*(X_n \leq x) = F_X(x) = P(X \leq x)$
$\Rightarrow \lim_{n\to\infty} P_*(X_n \leq x) = \lim_{n\to\infty} P^*(X_n \leq x) = P(X \leq x)$

$\Rightarrow X_n \rightsquigarrow X$.