Solutions to Problem Set 3

6.5.8 Denote \( \pi_t \) as the coordinate projections \( x \mapsto x(t) \) on \( C[a, b] \), then 
\( \pi_t(x) = x(t) \);
First prove that \( \pi_t \) is continuous: \( \forall \{x_a\} \subset C[a, b], x_a \to x, |\pi_t x_a - \pi_t x| = |x_a(t) - x(t)| \leq \sup|x_a(t) - x(t)| = \|x_a(t) - x(t)\| \), hence \( \pi_t \) is continuous.
Therefore \( \forall \) open set \( O_a \) in \( \mathbb{R}, \pi_t^{-1}(O_a) \) is open; Since the open sets generate the Borel \( \sigma \)-field, \( \pi_t^{-1}(O_a) \) is contained in the Borel \( \sigma \)-field; Therefore, \( \pi_t^{-1}(O_a) \) is contained in the Borel \( \sigma \)-field generated by the uniform norm, \( \sigma_p \subset \sigma_c \).

Now we are trying to prove \( \sigma_c \subset \sigma_p \):
Denote \( B(x, r) = \{z : \|z - x\| \leq r\} \), i.e \( B(x, r) \) is closed balls in \( \sigma_c \), we can prove that \( B(x, r) = \bigcap \{z : |\pi_t x - \pi_t z| \leq r\} \), where \( t \) is rational number, \( t \in [a, b] \);
If \( z \in \bigcap_{t \in Q[a,b]} \{z : |\pi_t x - \pi_t z| \leq r\} \), here \( Q \) is the set of all rational numbers, then \( |\pi_t x - \pi_t z| \leq r \) for all \( t \in Q[a,b] \); Since \( Q \) is dense in \( \mathbb{R} \), then \( \forall t' \in [a, b], \forall \epsilon > 0, \exists t \in Q[a,b], s.t \|t - t'\| < \epsilon \), since \( x, z \in C[a, b], \forall \delta > 0, \exists t \in Q[a,b], \|t - t'\| < \epsilon, \) s.t \( |x(t') - x(t)| < \delta, |z(t') - z(t)| < \delta \); Therefore, \( |x(t') - z(t')| \leq |x(t') - x(t)| + |x(t) - z(t)| + |z(t) - z(t')| < r + 2\delta \), then \( \sup_{t \in [a,b]} |x(t') - z(t')| \leq r \), \( \|x - z\| \leq r \), we can get \( z \in B(x, r) \). Hence \( B(x, r) \supset \bigcap \{z : |\pi_t x - \pi_t z| \leq r\} \);
If \( z \in B(x, r) \) i.e \( \|z - x\| \leq r, \forall t \in [a, b], |z(t) - x(t)| \leq \sup_{t \in [a,b]} |x(t) - z(t)| = \|z - x\| \leq r \), thus we have \( \pi_t z - \pi_t x \leq r \) for all \( t \in [a, b] \), so \( z \in \bigcap \{z : |\pi_t x - \pi_t z| \leq r\}, t \in Q[a,b] \), Hence \( B(x, r) \subset \bigcap_{t \in Q[a,b]} \{z : |\pi_t x - \pi_t z| \leq r\} \);
Therefore, we have proved that \( B(x, r) = \bigcap_{t \in Q[a,b]} \{z : |\pi_t x - \pi_t z| \leq r\} \).
Notice that closed balls can generate open balls, therefore closed balls can generate Borel \( \sigma \)-field. Clearly, each open ball is contained in a closed ball, and each closed ball \( B(x, r) = \{z : \|z - x\| \leq r\} \) is contained in the open ball \( B(x, r + \epsilon) = \{z : \|z - x\| < r + \epsilon \}, \epsilon > 0\).
Then we get \( \sigma_p \supset \sigma_c \), and the desired result \( \sigma_p = \sigma_c \) follows.

6.5.13 (a) \[ |S^* - T_*| = |S^* - T^* + T^* - T_*| \leq |S^* - T^*| + |T^* - T_*| \leq |S - T^*| + T^* - T_* ; \]
\[ |S^* - T_*| = |S^* - S_* + S_* - T_*| \leq |S_* - T_*| + |S^* - S_*| = \|(-T)^* - (-S)^*\| + |S^* - S_*| \leq |S - T^*| + |S^* - S_*| \leq |S - T| + S^* - S_*; \]
\[ |S^* - T_*| \leq |S - T|^* + (S^* - S_*) \wedge (T^* - T_*) \]

(b). \[ |S - T|^* = (S - T)^* \vee (T - S)^* \leq [S^* + (-T)^*] \vee [(T^* + (-S)^*)] = (S^* - T_*) \vee (T^* - S_*) \]

From (a), we know \( S^* - T_* \leq |S - T|^* + (S^* - S_*) \wedge (T^* - T_*) \)

\( T^* - S_* \leq |T - S|^* + (T^* - T_*) \wedge (S^* - S_*) \)

\[ (S^* - T_*) \vee (T^* - S_*) \leq |S - T|^* + (S^* - S_*) \wedge (T^* - T_*) \]

Therefore, \( |S - T|^* \leq (S^* - T_*) \vee (T^* - S_*) \leq |S - T|^* + (S^* - S_*) \wedge (T^* - T_*) \)