Solutions to Problem Set 3

6.5.8 Denote π_t as the coordinate projections $x \mapsto x(t)$ on $C[a,b]$, then $\pi_t(x) = x(t)$;
First prove that π_t is continuous: $\forall \{x_\alpha\} \subset C[a,b], x_\alpha \to x$, $|\pi_t x_\alpha - \pi_t x| = |x_\alpha(t) - x(t)| \leq sup|x_\alpha(t) - x(t)| = \| x_\alpha(t) - x(t) \|$, Hence π_t is continuous.
Therefore $\forall$ open set O_α in $\Re$,$\pi_t^{-1}(O_\alpha)$ is open; Since the open sets generate the Borel σ-field, $\pi_t^{-1}(O_\alpha)$ is contained in the Borel σ-field;
Therefore, $\pi_t^{-1}(O_\alpha)$ is contained in the Borel σ-field generated by the uniform norm, $\sigma_p \subset \sigma_c$.
Now we are trying to prove $\sigma_c \subset \sigma_p$:
Denote $B(x,r) = \{z : \| z - x \| \leq r\}$, i.e $B(x,r)$ is closed balls in σ_c, we can prove that $B(x,r) = \bigcap_t \{z : |\pi_t x - \pi_t z| \leq r\}$, where t is rational number,$t \in [a,b]$;
If $z \in \bigcap_{t \in Q \bigcap [a,b]} \{z : |\pi_t x - \pi_t z| \leq r\}$, here Q is the set of all rational numbers, then $|\pi_t x - \pi_t z| \leq r$ for all $t \in Q \bigcap [a,b]$;Since Q is dense in $\Re$, then $\forall t' \in [a,b], \forall \epsilon > 0, \exists t \in Q \bigcap [a,b]$,s.t $|t - t'| < \epsilon$, since $x, z \in C[a,b]$,then $\forall \delta > 0, \exists\, t \in Q \bigcap [a,b], |t - t'| < \epsilon$, s.t $|x(t') - x(t)| < \delta, |z(t') - z(t)| < \delta$;Therefore,$|x(t') - z(t')| \leq |x(t') - x(t)| + |x(t) - z(t)| + |z(t) - z(t')| < r + 2\delta$, then $\sup_{t' \in [a,b]} |x(t') - z(t')| \leq r$,$\| x - z \| \leq r$, we can get $z \in B(x,r)$.Hence $B(x,r) \supset \bigcap_t \{z : |\pi_t x - \pi_t z| \leq r\}$;
If $z \in B(x,r)$ i.e $\| z - x \| \leq r$,$\forall t \in [a,b], |z(t) - x(t)| \leq \sup_{t \in [a,b]} |x(t) - z(t)| = \| z - x \| \leq r$,thus we have $\pi_t z - \pi_t x \leq r$ for all $t \in [a,b]$, so $z \in \bigcap_t \{z : |\pi_t x - \pi_t z| \leq r\}, t \in Q \bigcap [a,b]$, Hence $B(x,r) \subset \bigcap_{t \in Q \bigcap [a,b]} \{z : |\pi_t x - \pi_t z| \leq r\}$;
Therefore,we have proved that $B(x,r) = \bigcap_{t \in Q \bigcap [a,b]} \{z : |\pi_t x - \pi_t z| \leq r\}$.
Notice that closed balls can generate open balls, therefore closed balls can generate Borel σ-field. Clearly, each open ball is contained in a closed ball, and each closed ball $B(x,r) = \{z : \| z - x \| \leq r\}$ is contained in the open ball $B(x, r + \epsilon) = \{z : \| z - x \| < r + \epsilon, \epsilon > 0\}$.
Then we get $\sigma_p \supset \sigma_c$, and the desired result $\sigma_p = \sigma_c$ follows.

6.5.13 (a). $|S^* - T_*| = |S^* - T^* + T^* - T_*| \leq |S^* - T^*| + |T^* - T_*| \leq |S - T|^* + T^* - T_*$;
$|S^* - T_*| = |S^* - S_* + S_* - T_*| \leq |S_* - T_*| + |S^* - S_*| = |(-T)^* - (-S)^*| + S^* - S_* \leq |-T - (-S)|^* + S^* - S_* \leq |S - T|^* + S^* - S_*$;

$\Rightarrow |S^* - T_*| \le |S-T|^* + (S^* - S_*) \bigwedge (T^* - T_*)$
(b). $|S-T|^* = (S-T)^* \bigvee (T-S)^* \le [S^* + (-T)*] \bigvee [T^* + (-S)^*] = (S^* - T_*) \bigvee (T^* - S_*)$;
From (a), we know $S^* - T_* \le |S-T|^* + (S^* - S_*) \bigwedge (T^* - T_*), T^* - S_* \le |T-S|^* + (T^* - T_*) \bigwedge (S^* - S_*)$
$\Rightarrow (S^* - T_*) \bigvee (T^* - S_*) \le |S-T|^* + (S^* - S_*) \bigwedge (T^* - T_*)$
Therefore, $|S-T|^* \le (S^* - T_*) \bigvee (T^* - S_*) \le |S-T|^* + (S^* - S_*) \bigwedge (T^* - T_*)$