Solutions to Problem Set 1

2.4.1 Let $F(s,t) \equiv P\mathbf{1}\{X \le s, Y \le t\}$, $F_1(s) = F(s,\infty)$, and $F_2(t) = F(\infty,t)$. For each $\eta > 0$, choose $-\infty = s_1 < s_2 < \cdots < s_{k_1} = \infty$ and $-\infty = t_1 < t_2 < \cdots < t_{k_2} = \infty$ such that $F_1(s_j-) - F_1(s_{j-1}) < \eta$, for all $1 < j \le k_1$, and $F_2(t_j-) - F_2(t_{j-1}) < \eta$, for all $1 < j \le k_2$. This can be done so that both k_1 and k_2 are $\le 2 + 1/\eta$. Consider the set of brackets of the form (l,u), where $l(X,Y) = \mathbf{1}\{X \le s_{j-1}, Y \le t_{l-1}\}$ and $u(X,Y) = \mathbf{1}\{X < s_j, Y < t_l\}$, for $1 < j \le k_1$ and $1 < l \le k_2$. The number of such brackets is bounded by $(1+1/\eta)^2$ by construction. Note that for these brackets, $\|u - l\|_{P,r} < \eta^{1/r}$, for all $1 \le r < \infty$. Hence, for $\mathcal{F} = \{\mathbf{1}\{X \le s, Y \le t\} : s,t \in \mathbb{R}\}$, we have $N_{[]}(\epsilon, \mathcal{F}, L_1(P)) \le (1+1/\epsilon)^2$ and $N_{[]}(\epsilon, \mathcal{F}, L_2(P)) \le (1+1/\epsilon^2)^2$.

3.5.1 We need to make one more assumption that

$$\int \dot{\ell}_{\tilde\theta}\dot{\ell}'_{\tilde\theta} p_{\tilde\theta} d\mu \;\to\; \int \dot{\ell}_\theta \dot{\ell}'_\theta p_\theta d\mu, \text{ as } \|\tilde\theta - \theta\| \to 0. \tag{1}$$

We begin the proof by first showing that

$$H(\tilde\theta,\theta) \;\equiv\; \int \left(p_{\tilde\theta}^{1/2} - p_\theta^{1/2}\right)^2 d\mu \;\to\; 0, \text{ as } \|\tilde\theta - \theta\| \to 0. \tag{2}$$

By the mean value theorem, we have for some $\tilde\theta^*$ on the line segment between θ and $\tilde\theta$ that

$$\begin{aligned} H(\tilde\theta,\theta) &= \int \left(\dot{\ell}'_{\tilde\theta^*}(\tilde\theta - \theta) p_{\tilde\theta^*}^{1/2}\right)^2 d\mu \\ &\le \|\tilde\theta - \theta\|^2 \int \|\dot{\ell}_{\tilde\theta^*}\|^2 p_{\tilde\theta^*} d\mu \\ &\to 0, \end{aligned}$$

as $\|\tilde\theta - \theta\| \to 0$ by (1). Hence (2) follows.

Next, we show that (2) implies

$$\int |p_{\tilde\theta} - p_\theta| d\mu \;\to\; 0, \text{ as } \|\tilde\theta - \theta\| \to 0. \tag{3}$$

This follows immediately from the fact that

$$\begin{aligned}\int |p_{\tilde{\theta}} - p_\theta| d\mu &= \int \left| p_{\tilde{\theta}}^{1/2} - p_\theta^{1/2} \right| \left(p_{\tilde{\theta}}^{1/2} + p_\theta^{1/2} \right) d\mu \\ &\leq \sqrt{\int \left(p_{\tilde{\theta}}^{1/2} - p_\theta^{1/2} \right)^2 d\mu} \times \sqrt{\int \left(p_{\tilde{\theta}}^{1/2} + p_\theta^{1/2} \right)^2 d\mu} \\ &\leq 2 \left[H(\tilde{\theta}, \theta) \right]^{1/2}.\end{aligned}$$

Now, fix $h \in \mathbb{R}^k$, let the scalar sequence $t \to 0$, and set $dP = p_\theta d\mu$ and $dP_t = p_{\theta+th} d\mu$. We have by the mean value theorem that for some $\tilde{t} \in [0, t]$, and with $\tilde{\theta} \equiv \theta + \tilde{t}h$, the left-hand-side of expression (3.1) of page 37 of Kosorok (2008) is equal to

$$\begin{aligned}\int \left(h'\dot{\ell}_{\tilde{\theta}} p_{\tilde{\theta}}^{1/2} - h'\dot{\ell}_\theta p_\theta^{1/2} \right)^2 d\mu &= \int J_1 J_2 \left(h'\dot{\ell}_{\tilde{\theta}} p_{\tilde{\theta}}^{1/2} - h'\dot{\ell}_\theta p_\theta^{1/2} \right)^2 d\mu \\ &\quad + \int (1 - J_1 J_2) \left(h'\dot{\ell}_{\tilde{\theta}} p_{\tilde{\theta}}^{1/2} - h'\dot{\ell}_\theta p_\theta^{1/2} \right)^2 d\mu \\ &\equiv A_t + B_t,\end{aligned}$$

where $J_1 = \mathbf{1}\{\|\dot{\ell}_\theta\| \leq k_1\}$, $J_2 = \mathbf{1}\{\|\dot{\ell}_{\tilde{\theta}}\| \leq k_2\}$, and $0 < k_1, k_2 < \infty$ are scalars (to be chosen later).

Clearly,

$$\begin{aligned}A_t &\leq \int J_1 J_2 \left[h'(\dot{\ell}_{\tilde{\theta}} - \dot{\ell}_\theta) \right]^2 p_\theta d\mu + \int J_1 J_2 (h'\dot{\ell}_{\tilde{\theta}})^2 (p_{\tilde{\theta}}^{1/2} - p_\theta^{1/2})^2 d\mu \\ &\leq \|h\|^2 \int \|\dot{\ell}_{\tilde{\theta}} - \dot{\ell}_\theta\|^2 p_\theta d\mu + \|h\|^2 k_2^2 H(\tilde{\theta}, \theta) \\ &\to 0,\end{aligned}$$

where, in the second-to-last line, the first term goes to zero by assumption and the second term goes to zero by (2).

To facilitate proof that $B_t \to 0$, we first argue that

$$\int (1 - J_1 J_2) \dot{\ell}_{\tilde{\theta}} \dot{\ell}_{\tilde{\theta}}' p_{\tilde{\theta}} d\mu = \int (1 - J_1 J_2) \dot{\ell}_\theta \dot{\ell}_\theta' p_\theta d\mu + o(1). \qquad (4)$$

By applying (3) followed by bounded convergence,

$$\begin{aligned}\int J_1 J_2 \dot{\ell}_{\tilde{\theta}} \dot{\ell}'_{\tilde{\theta}} p_{\tilde{\theta}} d\mu &= \int J_1 J_2 \dot{\ell}_{\tilde{\theta}} \dot{\ell}'_{\tilde{\theta}} p_{\theta} d\mu + o(1) \\ &= \int J_1 J_2 \dot{\ell}_{\theta} \dot{\ell}'_{\theta} p_{\theta} d\mu + o(1).\end{aligned}$$

Combining this with assumption (1), (4) follows. Hence

$$\begin{aligned}B_t &\leq 4 \int (1 - J_1 J_2)(h'\dot{\ell}_\theta)^2 p_\theta d\mu + o(1) \\ &\leq 4 \int (1 - J_1)(h'\dot{\ell}_\theta)^2 p_\theta d\mu + 4 \int J_1(1 - J_2)(h'\dot{\ell}_\theta)^2 p_\theta d\mu + o(1) \\ &\leq 4\|h\|^2 \int \mathbf{1}\{\|\dot{\ell}_\theta\| > k_1\} \|\dot{\ell}_\theta\|^2 p_\theta d\mu + 4\|h\|^2 k_1^2 P(\|\dot{\ell}_{\tilde{\theta}}\| > k_2) + o(1) \\ &\equiv 4\|h\|^2 C_1(k_1) + 4\|h\|^2 k_1^2 C_2(k_2) + o(1).\end{aligned}$$

Since k_1 and k_2 are arbitrary, we can, for any $\epsilon > 0$, first choose $k_1 \geq 1$ large enough so that $C_1(k_1) \leq \epsilon/(8\|h\|^2)$ and then pick k_2 large enough so that $C_2(k_2) \leq \epsilon/(8\|h\|^2 k_1^2)$, and thus $B_t \leq \epsilon + o(1)$. Hence $B_t \to 0$ since ϵ was also arbitrary. Thus $A_t + B_t \to 0$, and the desired result follows.$\square$