# A Monte Carlo Method for Obtaining the Null Distribution of Function-indexed Log-rank Statistics

Department of Biostatistics Technical Report 127

Michael R. Kosorok [1]

## ABSTRACT

In this paper, we develop a Monte Carlo method for accurately obtaining p-values for the function-indexed statistics described in Kosorok and Lin (1998).

[1]Michael R. Kosorok is Assistant Professor, Departments of Statistics and Biostatistics, University of Wisconsin, Madison, WI 53706. The author would like to thank Chin-Yu Lin for several insightful comments. This research was funded by grant CA75142 from the National Cancer Institute.

## 1. INTRODUCTION

The complexity of the limiting distributions of the two-sample function-indexed statistics described in Kosorok and Lin (1998) preclude obtaining p-values through analytical means for many function-indexed test procedures. In this paper, we describe a Monte-Carlo approach to obtaining distributional properties of the proposed statistics. This approach is related to the method used by Lin, Wei, and Ying (1993) to simulate the null distribution of partial sums of martingale residuals from a Cox proportional hazards regression model and to the method used by Su and Wei (1991) to simulate the null distribution of partial sums of residuals from generalized linear models. The notation we use is as given in Kosorok and Lin (1998), Sections 1 through 3, and Kosorok (1998), Section 4.

## 2. THE MONTE CARLO METHOD

Let $n = n_1 + n_2$ be the total sample size, for samples 1 and 2, and let $Q$ be the desired number of Monte Carlo replicates. We first obtain $nQ$ standard normal random deviates, $Z_{ijq}$, $i = 1 \ldots n_j$, $j = 1, 2$, $q = 1 \ldots Q$, and construct the corresponding artificial martingales

$$\widetilde{M}^n_{ijq}(t) \quad = \quad \int_0^{t\wedge\tau_n} \left\{1 - \frac{1}{\overline{Y}_j(s)}\right\}^{-1/2} Z_{ijq} \left\{dN_{ij}(s) - Y_{ij}(s)\frac{d\overline{N}_j(s)}{\overline{Y}_j(s)}\right\},$$

where $\tau_n \equiv \sup\{t : \overline{Y}_1(t) \wedge \overline{Y}_2(t) > 1\}$. For $q = 1 \ldots Q$, we then construct

$$\widetilde{Z}^{qn}_f(t) \quad = \quad n^{-1/2}\left\{V^n_{ff}(\infty)\right\}^{-1/2} \int_0^t f\left(\boldsymbol{b}^n(s)\right) \frac{\overline{Y}_1(s)\overline{Y}_2(s)}{\overline{Y}_1(s) + \overline{Y}_2(s)} \left\{\frac{d\overline{M}^{qn}_1(s)}{\overline{Y}_1(s)} - \frac{d\overline{M}^{qn}_2(s)}{\overline{Y}_2(s)}\right\},$$

where

$$\overline{M}^{qn}_j(t) = \sum_{i=1}^{n_j} \widetilde{M}^n_{ijq}(t),$$

for $j = 1, 2$.

As we show in the theorem given in the next section, for any $t \in (0, \infty]$, this collection of $Q$ artificial realizations restricted to the time interval $[0, t]$ converges weakly on

$\{\mathbf{A}(\mathbf{H},\mathbf{D}[0,t]),a\}^Q$—conditional on the failure time data from the two samples—to $Q$ independent and identically distributed replicates of the asymptotic null distribution of $Z_f^n(\cdot)$. As defined in Kosorok and Lin (1998), the metric $a$, for any $x,y\in\mathbf{A}(\mathbf{H},\mathbf{D}[0,t])$, is

$$a(x,y)\equiv\sup_{f\in\mathbf{H}} d(x(f),y(f)),$$

where $d$ is a bounded and complete version of the Skorohod metric on $\mathbf{D}[0,t]$.

With this weak convergence result, we can obtain the null distribution of continuous mappings of $Z_f^n(\cdot)$, such as those given in result (ii) of Theorem 1 of Kosorok and Lin (1998). Regarding the computational algorithm, since additional sorting beyond that needed to compute $\overline{N}_j$ and $\overline{Y}_j$, for $j=1,2$, is not required, the computation rate is of order $O(n)$ for each "artificial martingale" replication; whereas sorting would be required for each replication from either a bootstrap or random permutation approach and the corresponding computation rate would therefore be of order $O(n\log n)$ at best. In addition to being computationally faster than either the bootstrap or random permutation approach, the artificial martingale approach is valid even when the censoring distributions for the two samples differ.

## 3. THE MAIN THEOREM AND PROOF

Theorem 1 *Suppose model 1 of Kosorok and Lin (1998) obtains with the corresponding sequence of histories* $\{\mathcal{F}^n,n\geq 1\}$. *Let* $\mathbf{H}$ *be either equal to or a closed subset of* $\mathbf{G}_r^+(K)$, *for some choice of* $r<\infty$ *and* $K<\infty$. *Also let* $\left\{\boldsymbol{b}^n=(b_1^n,\ldots,b_r^n)^T,n\geq 1\right\}$, *where* $\left\{b_j^n,n\geq 1\right\}\in\mathbf{B}\left(\{\mathcal{F}^n,n\geq 1\}\right)$ *for* $j=1\ldots r$ *and* $\mathbf{B}\left(\{\mathcal{F}^n,n\geq 1\}\right)$ *is as given in Definition 2 of Kosorok (1998). Then, provided that* $inf_{f\in\mathbf{H}}U_{ff}(\infty)>0$ *under submodel (L), or* $inf_{f\in\mathbf{H}}V_{ff}(\infty)>0$ *under submodel (F),*

*(i) For any* $t\in(0,\infty]$, *the collection* $\left\{\widetilde{Z}_f^{qn}(\cdot),f\in\mathbf{H},q=1\ldots Q\right\}$ *converges weakly on* $\{\mathbf{A}(\mathbf{H},\mathbf{D}[0,t]),a\}^Q$ *to a collection of* $Q$ *independent Gaussian processes,* $\left\{Z_f^q(\cdot),f\in\mathbf{H},q=1\ldots Q\right\}$,

*such that each $Z_f^q$ has mean 0 and independent increments with covariance function $C_{fg}^0(\cdot)$ under submodel (L) or $C_{fg}(\cdot)$—defined in Kosorok (1998) and closely related to $\tilde{C}_{fg}(\cdot)$—under submodel (F), where $\left\{Z_f^{qn}, n \geq 1\right\}$ and $Z_f^g$ are restricted to $[0,t]$;*

*(ii) Functionals of the form given in result (ii) of Theorem 1 of Kosorok and Lin (1998) converge weakly in the uniform topology on $\left\{\mathbf{A}(\mathbf{H}, \mathcal{R}^3)\right\}^Q$ to $Q$ independent realizations of the corresponding functionals applied to $Z_f^q(\cdot)$ given above; and*

*(iii) For $f, g \in \mathbf{H}$, $q = 1 \ldots Q$, $E\left[\widetilde{Z}_f^{qn}(s)\widetilde{Z}_g^{qn}(t)\right] = E\left[Z_f^{0n}(s \wedge \tau_n) Z_g^{0n}(t \wedge \tau_n)\right]$, $\forall n \geq 1$, where*

$$Z_f^{0n}(t) = \left\{V_{ff}^n(\infty)\right\}^{-1/2} n^{-1/2} \int_0^t f\left(\boldsymbol{b}^n(x)\right) \frac{\overline{Y}_1(x)\overline{Y}_2(x)}{\overline{Y}_1(x) + \overline{Y}_2(x)} \left\{\frac{d\overline{M}_1^n(x)}{\overline{Y}_1(x)} - \frac{d\overline{M}_2^n(x)}{\overline{Y}_2(x)}\right\};$$

*ie., up until the stopping time $\tau_n$, the small sample second moments of $\widetilde{Z}_f^{qn}(\cdot)$ and the null distribution of $Z_f^n(\cdot)$, $Z_f^{0n}(\cdot)$, coincide.*

**Proof of Theorem 1**. We will begin by utilizing an adaptation of the time-transform method of Gill (1980), used in the proof of his Theorem 4.2.1, to establish weak convergence in the Skorohod topology on $(\mathbf{D}[0,\infty])^{mQ}$ of finite collections of function-indexed Monte Carlo replicates. We will then utilize Theorem 2 of Kosorok (1998) to obtain the necessary tightness on $\{\mathbf{A}(\mathbf{H}, \mathbf{D}[0,\infty])\}^Q$ and the proof of part (i) will be complete. Before presenting the proof, we need to define the filtration

$$\widetilde{\mathcal{F}}_t^n = \sigma\left\{\mathcal{F}_t^n, Z_{ijq}, i = 1 \ldots n_j; j = 1, 2; q = 1 \ldots Q\right\}.$$

This is simply the usual filtration, as defined in Section 2.1, augmented by the Gaussian random variables used to generate the Monte Carlo replicates. We will also need the following lemma:

LEMMA 1 *Let $X_i^n$, $i = 1 \ldots n$, be a triangular array of real random variables (possibly dependent) such that $E\left[(X_i^n)^2\right] \leq \kappa$ for some $\kappa < \infty$ and all $1 \leq i \leq n \leq \infty$. Let*

*$\{Y_i^n(t), t \in T\}$ be a triangular array of monotone stochastic processes, where $T$ is a closed subset of $\mathcal{R}$ and*

$$\sup_{t\in T}\left|n^{-1}\sum_{i=1}^{n} Y_i^n(t) - \zeta(t)\right| \to 0,$$

*in probability, as $n \to \infty$, for some left- or right- continuous, bounded $\zeta(t)$. If*

$$\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} X_i^n Y_i^n(t) \to \xi(t),$$

*in probability, for some real function $\xi(t)$ and each $t \in T$, then the convergence is also uniform over $t \in T$.*

Proof. Without loss of generality, assume $\zeta(t)$ is right-continuous. For each $\epsilon > 0$, construct a finite partition $R_\epsilon(T)$ of $T$ consisting of non-overlapping intervals of the form $[a, b)$ or $[a, b]$, for $a < b$, such that

$$\sup_{p\in R_\epsilon(T)} \sup_{s,t\in p} |\zeta(t) - \zeta(s)| < \epsilon.$$

For each $p \in R_\epsilon(T)$, let $L(p)$ be the left endpoint and $U(p)$ be the right endpoint. Let $E_\epsilon(T)$ be all the endpoints of all intervals in $R_\epsilon(T)$. Now,

$$\begin{aligned}
\sup_{t\in T}\left|n^{-1}\sum_{i=1}^{n} X_i^n Y_i^n(t) - \xi(t)\right| &\le \sup_{t\in E_\epsilon(T)}\left|n^{-1}\sum_{i=1}^{n} X_i^n Y_i^n(t) - \xi(t)\right| \\
&\quad + \sup_{p\in R_\epsilon(T)} \sup_{s,t\in p}\left|n^{-1}\sum_{i=1}^{n} X_i^n Y_i^n(t) - n^{-1}\sum_{i=1}^{n} X_i^n Y_i^n(s)\right| \\
&\quad + \sup_{p\in R_\epsilon(T)} \sup_{s,t\in p} |\xi(t) - \xi(s)|;
\end{aligned}$$

however,

$$\begin{aligned}
&\sup_{s,t\in p}\left|n^{-1}\sum_{i=1}^{n} X_i^n Y_i^n(t) - n^{-1}\sum_{i=1}^{n} X_i^n Y_i^n(s)\right| \\
&\le \left(n^{-1}\sum_{i=1}^{n}\{X_i^n\}^2\right)^{1/2}\left|n^{-1}\sum_{i=1}^{n} Y_i^n(U(p)-) - n^{-1}\sum_{i=1}^{n} Y_i^n(L(p))\right|^{1/2}.
\end{aligned}$$

Thus

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\sup_{p\in R_\epsilon(T)}\sup_{s,t\in p}\left|n^{-1}\sum_{i=1}^{n} X_i^n Y_i^n(t) - n^{-1}\sum_{i=1}^{n} X_i^n Y_i^n(s)\right| = 0,$$

in probability, and the result follows.□

Proof of Theorem 1, continued. For any collection of $mQ$ bounded left-continuous step functions on $[0,\infty]$, $\{c_{kq}(\cdot), k=1\ldots m, q=1\ldots Q\}$, let

$$C_q^n(t)=\sum_{k=1}^{m} c_{kq}(t) f_k(\boldsymbol{b}^n(t)),$$

where $f_k \in \mathbf{H}$, $k=1\ldots m$, and $m<\infty$. Define the process $X^n(t)\equiv X_1^n(t)+X_2^n(t)$, where

$$\begin{aligned} X_j^n(t) &\equiv \sum_{i=1}^{n_j}\int_0^t H_{ij}(s)dM_{ij}(s), \\ H_{ij}(t) &\equiv n^{-1/2}\frac{I_{\{\overline{Y}_1(t)\wedge\overline{Y}_2(t)>1\}}}{\overline{Y}_j(t)}\left\{1-\frac{1}{\overline{Y}_j(t)}\right\}^{-1/2}\frac{\overline{Y}_1(t)\overline{Y}_2(t)}{\overline{Y}_1(t)+\overline{Y}_2(t)} \\ &\quad \times\sum_{q=1}^{Q} C_q^n(t)\left\{Z_{ijq}-\frac{\sum_{l=1}^{n_j} Z_{ljq}Y_{ij}(t)}{\overline{Y}_j(t)}\right\}, \end{aligned}$$

and where

$$M_{ij}(t)\equiv N_{ij}(t)-\int_0^t Y_{ij}(s)d\Lambda_j^n(s),$$

$i=1\ldots n_j$, $j=1,2$. After a little algebra, one can show that

$$X^n(t)=\int_0^t C_q^n(s)\left\{\frac{d\overline{M}_1^{qn}(s)}{\overline{Y}_1(s)}-\frac{d\overline{M}_2^{qn}(s)}{\overline{Y}_2(s)}\right\},$$

where $\overline{M}_j^{qn}$, for $j=1,2$, are as defigned in Section 2 above. Under submodels (L) or (F), we can enumerate all points of discontinuity for both $\Lambda_1^n(t)$ and $\Lambda_2^n(t)$, for all $n\geq 1$, in a single sequence $T=\{t_1,t_2,\ldots\}$ such that for any $t<\infty$, the number of points of discontinuity less than or equal to $t$ is finite. Choose $\delta_k>0$, $k=1,2,\ldots$ such that $\sum_{k=1}^{\infty}\delta_k<\infty$.

Define the time transformation $\psi^*:[0,\infty]\mapsto[0,\infty]$ by

$$\psi^*(t)=t+\sum_{k:t_k\leq t}\delta_k.$$

Using $\mathbf{I}$ and $u$ as defined at the end of Section 2.1 of Kosorok and Lin (1998), let $\mathbf{I}^*=[0,\psi^*(u-))$ if $u\notin\mathbf{I}$ or $\mathbf{I}^*=[0,\psi^*(u)]$ if $u\in\mathbf{I}$, and let $u^*=\sup\mathbf{I}^*$. For $i=1\ldots n_j$, $j=1,2$, define the time-transformed processes $N_{ij}^*$, $Y_{ij}^*$, $M_{ij}^*$, and $H_{ij}^*$ as follows. Whenever

$t^* = \psi^*(t)$ for some $t$, let $N_{ij}^*(t^*) = N_{ij}(t)$, $Y_{ij}^*(t^*) = Y_{ij}(t)$, $M_{ij}^*(t^*) = M_{ij}(t)$, and $H_{ij}^*(t^*) = H_{ij}(t)$. On the intervals $[\psi^*(t_k-), \psi^*(t_k))$, for $t_k \in T$, $k = 1, 2, \ldots$, allow $N_{ij}^*(t^*)$, conditional on $Y_{ij}(t_k)$, to make a single jump at the point $R_{ijk}$ with probability $Y_{ij}(t_k)\Delta\Lambda_j^n(t_k)$, where $R_{ijk}$ is an independent random variable uniformly distributed on $(\psi^*(t_k-), \psi^*(t_k))$. Also for $t^* \in [\psi^*(t_k-), \psi^*(t_k))$, define $Y_{ij}^*(t^*) = Y_{ij}(t_k)$, $H_{ij}^*(t^*) = H_{ij}(t_k)$, and

$$M_{ij}^*(t^*) = M_{ij}(t_k-) + N_{ij}^*(t^*) - N_{ij}(t_k-) - Y_{ij}(t_k) I_{\{R_{ijk} \le t^*\}} \Delta\Lambda_j^n(t_k),$$

for $k = 1, 2, \ldots$.

If we define

$$\mathcal{E}_{t^*}^{*n} \equiv \sigma\left\{R_{ijk}, \text{for all } k : \psi^*(t_k-) \le t^*; N_{ij}^*(s^*), s^* \le t^*; i = 1 \ldots n_j; j = 1, 2\right\}$$

and

$$\mathcal{F}_{t^*}^{*n} \equiv \begin{cases} \sigma\left\{\widetilde{\mathcal{F}}_t^n, \mathcal{E}_{t^*}^{*n}\right\}, \text{ if } t^* = \psi^*(t), \\ \\ \sigma\left\{\widetilde{\mathcal{F}}_{t-}^n, \mathcal{E}_{t^*}^{*n}\right\}, \text{ if } \psi^*(t-) \le t^* < \psi^*(t), \end{cases}$$

it is not difficult to show that $M_{ij}^*(t^*)$, $i = 1 \ldots n_j$, $j = 1, 2$, are square integrable $\mathcal{F}_{t^*}^{*n}$-martingales, with $H_{ij}^*(t^*)$ and $Y_{ij}^*(t^*)$ being $\mathcal{F}_{t^*}^{*n}$-predictable.

For $j = 1, 2$, we now define

$$X_j^*(t^*) = \sum_{i=1}^{n_j} \int_0^{t^*} H_{ij}^*(s) dM_{ij}^*(s)$$

and obtain the predictable covariations $\left\langle M_{ij}^*, M_{lr}^*\right\rangle(\cdot) = 0$, for $i \ne l$ or $j \ne r$; $\left\langle M_{ij}^*, M_{ij}^*\right\rangle(t^*) = \left\langle M_{ij}, M_{ij}\right\rangle(t)$, for $t^* = \psi^*(t)$; and

$$\left\langle M_{ij}^*, M_{ij}^*\right\rangle(t^*) = \left\langle M_{ij}, M_{ij}\right\rangle(t_k-) + Y_{ij}(t_k) I_{\{R_{ijk} \le t^*\}} \left(1 - \Delta\Lambda_j^n(t_k)\right) \Delta\Lambda_j^n(t_k),$$

for $\psi^*(t_k-) \le t^* < \psi^*(t_k)$, $t_k \in T$. By Theorem 2.4.3 of Fleming and Harrington (1991), $X_1^*$ and $X_2^*$ have predictable covariation zero while $X_j^*$, $j = 1, 2$ has predictable variation

$$\left\langle X_j^*, X_j^*\right\rangle(t^*) \quad = \quad \sum_{i=1}^{n_j} \int_0^{t^*} \left[H_{ij}^*(s)\right]^2 d\left\langle M_{ij}^*, M_{ij}^*\right\rangle(s)$$

$$
= \begin{cases} \sum_{i=1}^{n_j} \int_0^t H_{ij}^2(s) d\left\langle M_{ij}, M_{ij}\right\rangle(s), \text{ if } t^* = \psi^*(t), \\ \sum_{i=1}^{n_j} \int_0^{t_k-} H_{ij}^2(s) d\left\langle M_{ij}, M_{ij}\right\rangle(s) \\ + \sum_{i=1}^{n_j} H_{ij}^2(t_k) Y_{ij}(t_k) I_{\{t^* \le R_{ijk}\}} \left(1 - \Delta\Lambda_j^n(t_k)\right) \Delta\Lambda_j^n(t_k), \\ \text{if } \psi^*(t_k-) \le t^* < \psi^*(t_k). \end{cases}
$$

It is not difficult to show that

$$
\lim_{n\to\infty} \sup_{s\in[0,t]} \left| \sum_{i=1}^{n_j} H_{ij}^2(t) Y_{ij}(t) - h(t) \right| = 0,
$$

in probability, for some $h(t)$ and all $t \in \mathbf{I}$; hence by Lemma 1,

$$
\begin{aligned}
\lim_{n\to\infty} \left\langle X_j^*, X_j^*\right\rangle(t^*) &= \begin{cases} \int_0^t h(s) d\Lambda_j(s), \text{ if } t^* = \psi^*(t), \\ \int_0^{t_k-} h(s) d\Lambda_j(s) + h(t_k) \frac{t^* - \psi^*(t_k-)}{\delta_k} \left(1 - \Delta\Lambda_j(t_k)\right) \Delta\Lambda_j(t_k), \\ \text{if } \psi^*(t_k-) \le t^* < \psi^*(t_k), \end{cases} \\
&\equiv W(t^*), \qquad (1)
\end{aligned}
$$

in probability, for each $t^* \in \mathbf{I}^*$, where $\Lambda_1(\cdot) = \Lambda_2(\cdot) = \Lambda_0(\cdot)$ under submodel (L).

If $u \notin \mathbf{I}$, then

$$
\begin{aligned}
\left\langle X_j^*, X_j^*\right\rangle(u^*) - \left\langle X_j^*, X_j^*\right\rangle(t^*) &= \int_{t^*}^{u^*} \left[H_{ij}^*(s)\right]^2 d\left\langle M_{ij}^*, M_{ij}^*\right\rangle(s) \\
&\le \int_{t-}^{u} H_{ij}^2(s) d\left\langle M_{ij}, M_{ij}\right\rangle(s),
\end{aligned}
$$

where $t$ is such that $\psi^*(t-) \le t^* < \psi^*(t)$. However,

$$
\begin{aligned}
& E\left[ \int_{t-}^{u} H_{ij}^2(s) d\left\langle M_{ij}, M_{ij}\right\rangle(s) \middle| \widetilde{\mathcal{F}}_u^n \right] \\
&= \int_{t-}^{u} I_{\{\overline{Y}_1(s) \wedge \overline{Y}_2(s) > 1\}} \left\{ \sum_{q=1}^{Q} \left[C_q^n(s)\right]^2 \right\} \left\{ 1 - \frac{\overline{Y}_j(s)}{\overline{Y}_1(s) + \overline{Y}_2(s)} \right\}^2 \\
&\quad \times \frac{\overline{Y}_j(s)}{n} \left(1 - \Delta\Lambda_j^n(s)\right) d\Lambda_j^n(s) \\
&\le \kappa \int_{t-}^{u} I_{\{\overline{Y}_1(s) \wedge \overline{Y}_2(s) > 1\}} \left\{ 1 - \frac{\overline{Y}_j(s)}{\overline{Y}_1(s) + \overline{Y}_2(s)} \right\}^2 \frac{\overline{Y}_j(s)}{n} \left(1 - \Delta\Lambda_j^n(s)\right) d\Lambda_j^n(s), \qquad (2)
\end{aligned}
$$

for some $\kappa < \infty$. Using methods similar to those used in the proof of Theorem 3 of Kosorok (1998), it is now straightforward to show that the limit as $t \uparrow u$ of the lim sup over $n$ of (2) goes to zero in probability and that

$$\lim_{t \uparrow u} \int_{t-}^{u} h(s) \left(1 - \Delta\Lambda_j(s)\right) d\Lambda_j(s) = 0.$$

These same methods can also show that if $u < \infty$, $h(t) = 0$ and $\overline{Y}_1(t) \wedge \overline{Y}_2(t) = 0$ almost surely for all $t > u$; and we therefore have that $\lim_{n\to\infty} \left\langle X_j^*, X_j^* \right\rangle (t^*) = W(t^*)$, in probability, for every $t^* \in [0, \infty]$, where $W(t^*)$ is given in (1).

Let $X_j^{*\epsilon}(t^*)$ be the process containing all the jumps in $X_j^*(t^*)$ of size $\epsilon$ or greater. As a result of our previous construction,

$$X_j^{*\epsilon}(t^*) = \sum_{i=1}^{n_j} \int_0^{t^*} H_{ij}^*(s) I_{\left\{|H_{ij}^*(s)| > \epsilon\right\}} dM_{ij}^*(s),$$

and the quadratic variation of $X_j^{*\epsilon}$ is

$$\begin{aligned} \left\langle X_j^{*\epsilon}, X_j^{*\epsilon} \right\rangle (t^*) &= \sum_{i=1}^{n_j} \int_0^{t^*} \left[H_{ij}^*(s)\right]^2 I_{\left\{|H_{ij}^*(s)| > \epsilon\right\}} d\left\langle M_{ij}^*, M_{ij}^* \right\rangle (s) \\ &\leq \sum_{i=1}^{n_j} \int_0^{t} H_{ij}^2(s) I_{\{|H_{ij}(s)| > \epsilon\}} d\left\langle M_{ij}, M_{ij} \right\rangle (s), \end{aligned}$$

where $t$ is such that $\psi^*(t-) \leq t^* \leq \psi^*(t)$. However,

$$\begin{aligned} & E\left[\sum_{i=1}^{n_j} \int_0^t H_{ij}^2(s) I_{\{|H_{ij}(s)| > \epsilon\}} d\left\langle M_{ij}, M_{ij} \right\rangle (s)\right] \\ &\leq \sum_{i=1}^{n_j} \left(E\left[Y_{ij}(s) H_{ij}^4(s)\right]\right)^{1/2} \left(P\left[|H_{ij}(s)| > \epsilon\right]\right)^{1/2} \left(1 - \Delta\Lambda_j^n(s)\right) d\Lambda_j^n(s) \\ &\leq \epsilon^{-1} \sum_{i=1}^{n_j} \left(E\left[H_{ij}^4(s)\right]\right)^{1/2} \left(E\left[H_{ij}^2(s)\right]\right)^{1/2} \left(1 - \Delta\Lambda_j^n(s)\right) d\Lambda_j^n(s) \\ &\leq \epsilon^{-1} \sum_{i=1}^{n_j} \left(E\left[H_{ij}^4(s)\right]\right)^{3/4} \left(1 - \Delta\Lambda_j^n(s)\right) d\Lambda_j^n(s); \end{aligned}$$

and, conditional on $\mathcal{F}_t^n$, $H_{ij}(t)$ is normally distributed with mean 0 and variance bounded by $4n^{-1} \sum_{q=1}^{Q} \left[C_q^n(t)\right]^2$, and thus $E\left[H_{ij}^4(t) \,|\mathcal{F}_t^n\right] \leq 12\kappa^2 n^{-2}$, for some $\kappa < \infty$. Hence,

$\lim_{n\to\infty} \left\langle X_j^{*\epsilon}, X_j^{*\epsilon}\right\rangle(t^*) = 0$, in probability, for all $t^* \in \mathbf{I}^*$. Since

$$\left\langle X_j^{*\epsilon}, X_j^{*\epsilon}\right\rangle(t^*) \leq \left\langle X_j^{*}, X_j^{*}\right\rangle(t^*),$$

we have that $\lim_{n\to\infty} \left\langle X_j^{*\epsilon}, X_j^{*\epsilon}\right\rangle(t^*) = 0$, in probability, for all $t^* \in [0,\infty]$.

Rebolledo's theorem (see, for example, Theorem II.5.1 of Andersen, et al., 1992) combined with the version of the Cramer-Wold device given as Lemma C.3.1 of Fleming and Harrington (1991) now yields weak convergence in the Skorohod topology on $(\mathbf{D}[0,\infty])^{mQ}$ of collections of the form $\left\{\widetilde{Z}_1^{*q}(f,\cdot) - \widetilde{Z}_2^{*q}(f,\cdot), f \in \mathbf{h}, q = 1\ldots Q\right\}$, where $\mathbf{h} = \{f_1, f_2, \ldots, f_m\}$,

$$\widetilde{Z}_j^{*q}(f,t^*) \equiv n^{-1/2}\sum_{i=1}^{n_j}\int_0^{t^*} f\left(\boldsymbol{b}^{*n}(s)\right)\frac{\overline{Y}_1^*(s)\overline{Y}_2^*(s)}{\overline{Y}_1^*(s)+\overline{Y}_2^*(s)}dM_{ij}^*(s),$$

and $\boldsymbol{b}^{*n}(t^*) \equiv \boldsymbol{b}^n(t)$ for $t$ such that $\psi^*(t-) \leq t^* \leq \psi^*(t)$, to $Q$ independent and identically distributed replicates of a mean zero Gaussian process with independent increments and covariance function

$$V_{fg}^*(t^*) \equiv \begin{cases} V_{fg}(t), \text{ if } \psi^*(t) = t^*, \\ V_{fg}(t_k-) + \frac{t^*-\psi^*(t_k-)}{\delta_k}\Delta V_{fg}(t_k), \text{ if } \psi^*(t_k-) \leq t^* < \psi^*(t_k), \end{cases}$$

for $f, g \in \mathbf{h}$, and where $\Lambda_1(\cdot) = \Lambda_2(\cdot) = \Lambda_0(\cdot)$ under submodel (L).

Since the above limiting process is continuous, the Skorohod construction, as given in Theorem 2.4.3 of Gill (1981), can now be used to show that there exists a probability space wherein the forgoing convergence is almost sure in the uniform topology on $(\mathbf{D}[0,\infty])^{mQ}$. By deleting the extra time intervals (ie., taking the inverse transformation of $\psi^*$) and returning to the original time scale, we obtain almost sure convergence in the uniform topology—and hence weak convergence in the Skorohod topology in the original probability space—of collections of the form $\left\{\widetilde{Z}_1^{q}(f,\cdot) - \widetilde{Z}_2^{q}(f,\cdot), f \in \mathbf{h}, q = 1\ldots Q\right\}$, where

$$\widetilde{Z}_j^{q}(f,\cdot) \equiv n^{-1/2}\sum_{i=1}^{n_j}\int_0^{(\cdot)} f\left(\boldsymbol{b}^{n}(s)\right)\frac{\overline{Y}_1(s)\overline{Y}_2(s)}{\overline{Y}_1(s)+\overline{Y}_2(s)}dM_{ij}(s),$$

to $Q$ independent and identically distributed replicates of a mean zero Gaussian process with independent increments and covariance function $V_{fg}(\cdot)$, for $f, g \in \mathbf{h}$, where $\Lambda_1(\cdot) = \Lambda_2(\cdot) =$

$\Lambda_0(\cdot)$ under submodel (L). Obviously, these same results also hold true for convergence restricted to the time interval $[0,t]$, on the appropriate product of $\mathbf{D}[0,t]$, for any $t \in (0,\infty]$.

After noting that for any $\widetilde{\mathcal{F}}_t^n$-predictable $h^n$ with $|h^n| \leq c$,

$$E\left[\left\{\int_0^t h^n(s)d\left[\widetilde{Z}_1^q(s) - \widetilde{Z}_2^q(s)\right]\right\}^2\right] \leq 2c^2,$$

for all $t \in [0,\infty]$, Theorem 2 of Kosorok (1998) followed by the version of Slutsky's theorem given as Corollary 3.3 in Chapter 3 of Ethier and Kurtz (1986) completes the proof of part (i). Part (ii) now follows by the Continuous Mapping Theorem. Part (iii) follows immediately by taking the expectation of $\widetilde{Z}_f^{qn}(s)\widetilde{Z}_g^{qn}(t)$ conditional on $\mathcal{F}_{s\vee t}^n$.$\Box$

## REFERENCES


Andersen, P. K., Borgan, O., Gill, R. D., and Keiding, N. (1992), *Statistical Models Based on Counting Processes*, New York: Spring-Verlag.

Ethier, S. N., and Kurtz, T. G. (1986), *Markov Processes*, New York: John Wiley and Sons.

Fleming, T. R., and Harrington, D. P. (1984), "Evaluation of Censored Survival Data Test Procedures Based on Single and Multiple Statistics," *Topics in Applied Statistics*, 97–123, New York: Marcel Dekker.

Gill, R. D. (1980), *Censoring and Stochastic Integrals*, Tract 124, Amsterdam: Mathematisch Centrum.

Kosorok, M. R. (1998), "Families of Weighted Log-rank Statistics as Function-indexed Stochastic Processes (version 1c)," Technical Report 98, Department of Biostatistics, University of Wisconsin, Madison, Wisconsin.

Kosorok, M. R., and Lin, C.-Y. (1998), "The Versatility of Function-indexed Weighted Log-rank Statistics (version 2b)," Technical Report 105, Department of Biostatistics, University of Wisconsin, Madison, Wisconsin.

Lin, D. Y., Wei, L. J., and Ying, Z. (1993), "Checking the Cox Model With Cumulative Sums of Martingale-based Residuals," *Biometrika* **80**, 557–581.

Su, J. G., and Wei, L. J. (1991), "A Lack-of-fit Test for the Mean Function in a Generalized Linear Model," *Journal of the American Statistical Association* **86**, 420–426.