Solution to Homework #9

Question 1 (2.25 of Lehmann & Casella)

$X_1, \dots, X_m \overset{iid}{\sim} U(0,\theta)$. $\quad Y_1, \dots, Y_n \overset{iid}{\sim} U(0,\theta')$

$\therefore f(x_i) = \frac{1}{\theta} I\{x_i < \theta\}$. $[x_i > 0]$

$\therefore \prod_{i=1}^{m} f(x_i) = \frac{1}{\theta^m} \prod_{i=1}^{n} I\{x_i < \theta\} = \frac{1}{\theta^m} \cdot I\{x_{(m)} < \theta\}$ where $X_{(m)}$ is the m-th order statistic. [all X_i's > 0].

Similarly, $g(y_i) = \frac{1}{\theta'} I\{y_i < \theta'\}$.

$\Rightarrow \prod_{i=1}^{n} g(y_i) = \frac{1}{\theta'} I\{y_{(n)} < \theta'\}$

$\therefore$ the joint likelihood $L = \frac{1}{\theta^m \theta'^n} \cdot I\{X_{(m)} < \theta\} \, I\{Y_{(n)} < \theta'\}$.

So, $(X_{(m)}, Y_{(n)})$ is complete sufficient for (θ, θ').

Now, $f_1(x_{(m)}) = m \left[F(x_{(m)})\right]^{m-1} f(x_{(m)})$.

$= \frac{m \cdot x_{(m)}}{\theta^m} \cdot I\{X_{(m)} < \theta\}$. $\quad X_{(m)} > 0$

$\therefore \mathcal{E}(X_{(m)}) = \int_0^{\theta} x \cdot \frac{m x^{m-1}}{\theta^m} \cdot dx = \frac{m}{m+1} \cdot \theta$

Similarly, $\cancel{\mathcal{E}(Y_{(n)})} \; \mathcal{E}(1/Y_{(n)}) = \int_0^{\theta'} \frac{1}{y} \cdot \frac{n y^{n-1}}{\theta'^n} \cdot dy = \frac{n}{n-1} \cdot \frac{1}{\theta'}$.

$\therefore \mathcal{E}\left(\frac{X_{(m)}}{Y_{(n)}}\right) = \mathcal{E}(X_{(m)}) \cdot \mathcal{E}(1/Y_{(n)}) = \frac{mn}{(m+1)(n-1)} \cdot \theta/\theta'$ [$\because X_{(m)}$ & $Y_{(n)}$ are indep.].

$\Rightarrow \mathcal{E}\left[\frac{(m+1)(n-1)}{mn} \cdot \frac{X_{(m)}}{Y_{(n)}}\right] = \frac{\theta}{\theta'}$

$\therefore \frac{(m+1)(n-1)}{mn} \cdot \frac{x_{(m)}}{y_{(n)}}$ is an unbiased estimate of θ/θ'. But it is also a function of the complete sufficient statistic $(x_{(m)}, y_{(n)})$.

$\Rightarrow$ it is the UMVUE of θ/θ'

Question 2 (3.23 of Lehmann & Casella)

(a) $X_1, \dots, X_n \overset{iid}{\sim}$ Poisson(λ)

$$\Rightarrow \prod_{i=1}^{n} f(x_i) = \frac{1}{\left(\prod_{i=1}^{n} x_i!\right)} \cdot e^{-n\lambda} \cdot e^{\left[\left(\sum_{i=1}^{n} x_i\right) \log \lambda\right]}$$

So, it is an exponential family with $\underset{\sim}{\eta} = \eta_1 = \log \lambda$. $\underset{\sim}{T} = t = \sum_{i=1}^{n} x_i$,

$h(\underset{\sim}{x}) = \dfrac{1}{\prod_{i=1} x_i!}$ & $C(\eta) = e^{-ne^{-\eta_1}}$

$\therefore$ $t = \sum_{i=1}^{n} x_i$ is a complete, sufficient statistic for η & hence for λ.

Let $\delta^* = \left(1 - \frac{b}{n}\right)^t$

$$\therefore \mathcal{E}(\delta^*) = \sum_{t=0}^{\infty} \left(1 - \frac{b}{n}\right)^t \cdot \frac{e^{-n\lambda} (n\lambda)^t}{t!} \quad \left[\because t = \sum_{i=1}^{n} x_i \sim \text{Poisson}(n\lambda)\right].$$

$$= e^{-n\lambda} \cdot \sum_{t=0}^{\infty} \frac{\left[n\lambda\left(1 - b/n\right)\right]^t}{t!}$$

$$= e^{-n\lambda} \cdot e^{n\lambda(1 - b/n)} = e^{-b\lambda}$$

$\therefore \delta^*$ is an unbiased estimate of $e^{-b\lambda}$. But it is also a function of the complete, sufficient statistic $\Rightarrow \delta^*$ is the UMVUE of $e^{-b\lambda}$.

(b) If $b > n$, $\left(1 - \frac{b}{n}\right) < 0 \Rightarrow \delta^* = \left(1 - \frac{b}{n}\right)^t < 0$ whenever t is odd. But, $e^{-b\lambda}$ is always positive.

Hence, we can obtain a negative estimate of a positive quantity. So, δ^* is not a reasonable estimate of $e^{-b\lambda}$ if $b > n$.

[Proved].

Question 3 (Q2 of Lecture Notes)

$$f_\theta(x) = \frac{\theta}{(1+x)^{\theta+1}} I_{\{x>0\}}.$$

$X_1, \ldots, X_n \overset{iid}{\sim} f_\theta$.

$P_\theta(\underset{\sim}{x}) = \prod_{i=1}^{n} f_\theta(x_i)$

$\therefore\ l_\theta = \log P_\theta(\underset{\sim}{x}) = n\log\theta - (\theta+1)\sum_{i=1}^{n}\log(1+x_i).$

$\therefore\ \frac{\partial(l_\theta)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\log(1+x_i).$ $\quad \frac{\partial^2 l_\theta}{\partial\theta^2} = -\frac{n}{\theta^2}.$

$\therefore$ the MLE of θ is given by $\hat{\theta}_n$ such that $\left.\frac{\partial l_\theta}{\partial\theta}\right|_{\theta=\hat{\theta}_n} = 0$

$\Rightarrow\ \hat{\theta}_n = \frac{n}{\sum_{i=1}^{n}\log(1+x_i)}.$

Now $I_n(\theta) = \mathcal{E}\left[-\frac{\partial^2 l_\theta}{\partial\theta^2}\right] = \frac{n}{\theta^2}.\ \Rightarrow I(\theta) = \frac{1}{n}I_n(\theta) = \frac{1}{\theta^2}.$

$\therefore\ I^{-1}(\theta) = \theta^2.$

Now, we know that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, I^{-1}(\theta)).$

$\Rightarrow\ \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \theta^2).$

(b) For any function $g(x)$, using Delta method, we get that

$\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{d} N(0, \{g'(\theta)\}^2 \cdot \theta^2).$

We want to find a function g, such that

$\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{d} N(0,1).$

$\Rightarrow\ \{g'(\theta)\cdot\theta\}^2 = 1.$

ie $g'(\theta) = \frac{1}{\theta} \Rightarrow g(\theta) = \log\theta + k$. (for any constant k).

$\therefore$ for $g(\theta) = \log\theta$ (taking $k=0$), we get

$$\sqrt{n}\,(\log(\hat{\theta}_n) - \log\theta) \xrightarrow{d} N(0,1).$$

(c) Now, using part (b) we get,

$$P\left[\,|\sqrt{n}\,(\log\hat{\theta}_n - \log\theta)| \leq Z_{\alpha/2}\right] \longrightarrow 1-\alpha$$

where $Z_{\alpha/2}$ is the $(1-\alpha/2)^{th}$ cutoff for $N(0,1)$ distribution.

$$\therefore P\left[\log\hat{\theta}_n - \frac{Z_{\alpha/2}}{\sqrt{n}} \leq \log\theta \leq \log\hat{\theta}_n + \frac{Z_{\alpha/2}}{\sqrt{n}}\right] \longrightarrow 1-\alpha$$

$$\Rightarrow P\left[e^{\left\{\log\hat{\theta}_n - \frac{Z_{\alpha/2}}{\sqrt{n}}\right\}} \leq \theta \leq e^{\left\{\log\hat{\theta}_n + Z_{\alpha/2}/\sqrt{n}\right\}}\right] \longrightarrow 1-\alpha$$

So, an approximate $(1-\alpha)$ confidence interval for θ based on (b) is $\left(e^{\{\log\hat{\theta}_n - Z_{\alpha/2}/\sqrt{n}\}},\ e^{\{\log\hat{\theta}_n + Z_{\alpha/2}/\sqrt{n}\}}\right)$.

Question 4 (Q3 of Lecture Notes)

(a) $X \sim$ standard exponential ie $f(x) = e^{-x} I\{x>0\}$

$Y/X=x \sim \text{Poisson}(\lambda x)$.

$\therefore P(Y=y) = \mathcal{E}[I\{Y=y\}] = \mathcal{E}[\mathcal{E}\{I_{\{Y=y\}}/X\}] = \mathcal{E}[P(Y=y/X)]$

$$\therefore P(Y=y) = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^y}{y!} e^{-x} dx$$

$$= \lambda^y \int_0^\infty \frac{e^{-(\lambda+1)x} \cdot x^y}{y!} dx$$

$$= \frac{\lambda^y}{(\lambda+1)^{y+1}} \int_0^\infty \frac{(\lambda+1) \cdot e^{-(\lambda+1)x} \cdot \{(\lambda+1)x\}^y}{y!} dx$$

$$= \frac{\lambda^y}{(\lambda+1)^{y+1}}$$ [Rest is integral over Gamma density = 1]

$\therefore Y \sim \text{Geometric}\left(\frac{1}{\lambda+1}\right)$.

Now, $\mathcal{E}(Y) = \mathcal{E}[\mathcal{E}(Y/X)] = \mathcal{E}[\lambda X] = \lambda \mathcal{E}(X) = \lambda$ $[\because \mathcal{E}(X)=1]$

$\text{var}(Y) = \mathcal{E}[\text{var}(Y/X)] + \text{var}[\mathcal{E}(Y/X)]$

$= \mathcal{E}[\lambda X] + \text{var}(\lambda X)$.

$= \lambda \mathcal{E}(X) + \lambda^2 \text{var}(X) = \lambda + \lambda^2$.

(b) The joint density of (X,Y) is given by

$$f(x,y) = e^{-(\lambda+1)x} \frac{(\lambda x)^y}{y!}, \quad x>0, \quad y = 1(1)\infty.$$

$\therefore \log[f(x,y)] = -(\lambda+1)x + y \log\lambda + h(x,y)$

[where $h(x,y)$ depends on x & y, but is indep. of λ]

$\therefore$ the log-likelihood is given by

$$l = \sum_{i=1}^{n} \log\left[f(x_i, y_i)\right] = -(\lambda+1)\sum_{i=1}^{n} x_i + \log\lambda\left(\sum_{i=1}^{n} y_i\right) + \sum_{i=1}^{n} h(x_i, y_i).$$

$$\therefore \frac{\partial^2 l}{\partial \lambda^2} = \frac{\partial}{\partial \lambda}\left[-\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i \Big/ \lambda\right] = -\sum_{i=1}^{n} y_i \Big/ \lambda^2$$

$$\therefore I_n(\lambda) = \mathcal{E}\left[-\frac{\partial^2 l}{\partial \lambda^2}\right] = \frac{n\,\mathcal{E}(Y)}{\lambda^2} = \frac{n}{\lambda}.$$

$$\therefore I(\lambda) = \frac{1}{\lambda}.$$

Now, for any unbiased estimate of λ, the Cramer-Rao lower bound is given by $1/I(\lambda) = \lambda$.

(c) First we determine $\hat{\lambda}_n$ based on the data $(X_1, Y_1), \ldots, (X_n, Y_n)$

$\therefore$ for the MLE we have

$$\left.\frac{\partial l}{\partial \lambda}\right|_{\lambda=\hat{\lambda}_n} = 0 \Rightarrow -\sum_{i=1}^{n} x_i + \frac{\sum_{i=1}^{n} y_i}{\hat{\lambda}_n} = 0 \Rightarrow \hat{\lambda}_n = \sum_{i=1}^{n} y_i \Big/ \sum_{i=1}^{n} x_i$$

$I_n(\lambda) = \frac{n}{\lambda}$ [from part (b)].

Now, consider $\check{\lambda}_n$ based on $Y_1, \ldots, Y_n$ only. $Y_i \sim \text{Geometric}\left(\frac{1}{\lambda+1}\right)$.

$$\therefore \log\left[P(Y=y)\right] = y\log\lambda - (y+1)\log(1+\lambda).$$

$\therefore$ the loglikelihood is given by

$$l_1 = \log\lambda\left(\sum_{i=1}^{n} y_i\right) - \log(1+\lambda)\left[\sum_{i=1}^{n} y_i + n\right].$$

$$\therefore \left.\frac{\partial l_1}{\partial \lambda}\right|_{\lambda=\check{\lambda}_n} = 0 \Rightarrow \check{\lambda}_n = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\frac{\partial^2 l_1}{\partial \lambda^2} = -\sum_{i=1}^{n} y_i \Big/ \lambda^2 + \frac{\sum_{i=1}^{n} y_i + n}{(1+\lambda)^2}. \quad \therefore I_2(\lambda) = \mathcal{E}\left[-\frac{\partial^2 l_1}{\partial \lambda^2}\right] = \frac{n}{\lambda(1+\lambda)}.$$

$\therefore$ the asymptotic relative efficiency of $\check{\lambda}_n$ vs $\hat{\lambda}_n$ is given by

$$\lim_{n\to\infty}\left[\frac{1/\text{var}(\check{\lambda}_n)}{1/\text{var}(\hat{\lambda}_n)}\right] = \frac{I_2(\lambda)}{I_n(\lambda)} = \frac{n}{\lambda(1+\lambda)}\cdot\frac{\lambda}{n} = \frac{1}{1+\lambda}.$$