Solution to Homework #8

Question 1 (Q1 of pg 104).

$X_1, \dots, X_n \overset{iid}{\sim}$ Poisson (λ).

So, $f(x_1, \dots, x_n) = \prod_{i=1}^{n} f(x_i) = \dfrac{e^{-n\lambda} \cdot \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} (x_i!)} = h(\underset{\sim}{x})\, c(\lambda) \exp\left[\left(\sum_{i=1}^{n} x_i\right)\log\lambda\right]$

where $h(\underset{\sim}{x}) = \dfrac{1}{\prod_{i=1}^{n}(x_i!)}$, $c(\lambda) = e^{-n\lambda}$, $T(\underset{\sim}{x}) = \sum_{i=1}^{n} x_i$. $\eta(\lambda) = \log\lambda$.

Hence, this is an exponential family with $T(\underset{\sim}{x}) = \sum_{i=1}^{n} x_i$ as the complete sufficient statistic. [Let $T = \sum_{i=1}^{n} x_i$].

Now, if we can find an unbiased estimate of θ^k using T, then it will be its required UMVUE.

Now consider $V = T(T-1)(T-2)\cdots(T-\overline{k-1})$.

$$\therefore\ \mathcal{E}(V) = \sum_{t=0}^{\infty} t(t-1)\cdots(t-\overline{k-1})\, e^{-n\lambda} \cdot \frac{(n\lambda)^t}{t!} \qquad [\because T \sim \text{Poisson}(n\lambda)]$$

$$= \sum_{t=k}^{\infty} e^{-n\lambda} \cdot \frac{(n\lambda)^t}{(t-k)!}$$

$$= (n\lambda)^k \cdot \sum_{z=0}^{\infty} e^{-n\lambda} \cdot \frac{(n\lambda)^z}{z!} \qquad [z = t-k]$$

$$= (n\lambda)^k \cdot 1 \qquad [\text{sum of Poisson } (n\lambda) \text{ probabilities}]$$

$$= n^k \lambda^k.$$

$$\therefore\ \mathcal{E}\left(\frac{1}{n^k} V\right) = \lambda^k.$$

$\therefore$ the required UMVUE of λ^k is given by

$$\frac{1}{n^k} \cdot V = \frac{1}{n^k} \left(\sum_{i=1}^{n} x_i\right) \cdot \left(\sum_{i=1}^{n} x_i - 1\right) \cdots \left(\sum_{i=1}^{n} x_i - \overline{k-1}\right).$$

Question 2 (Q2 of pg 104)

$X_i = \alpha + \beta t_i + \varepsilon_i$ where $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$, $i = 1(1)n$.

$\Rightarrow X_i \sim N(\alpha + \beta t_i, \sigma^2)$.

To obtain the LSE of α & β, we have to minimize

$T = \sum_{i=1}^{n} (X_i - \alpha - \beta t_i)^2$ wrt α & β.

$$\frac{\partial T}{\partial \alpha} = -2\left(\sum_{i=1}^{n} x_i - n\alpha - \beta \sum_{i=1}^{n} t_i\right).$$

$$\frac{\partial T}{\partial \beta} = -2\left(\sum_{i=1}^{n} x_i t_i - \alpha \sum t_i - \beta \sum_{i=1}^{n} t_i^2\right).$$

$\therefore$ to obtain $\hat{\alpha}$ & $\hat{\beta}$ we solve $\frac{\partial T}{\partial \alpha} = 0$ & $\frac{\partial T}{\partial \beta} = 0$. to obtain

$$\hat{\beta} = \frac{S_{xt}}{S_{tt}} \quad \left[\begin{array}{l} S_{xt} = \frac{1}{n}\sum_{i=1}^{n} x_i t_i - \left(\frac{1}{n}\sum x_i\right)\left(\frac{1}{n}\sum t_i\right) \rightarrow \text{sample cov. of } x \text{ \& } t \\ S_{tt} = \frac{1}{n}\sum_{i=1}^{n} t_i^2 - \left(\frac{1}{n}\sum t_i\right)^2 \rightarrow \text{sample var. of } t \end{array} \right]$$

$$\hat{\alpha} = \bar{x} - \hat{\beta}\bar{t} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \frac{S_{xt}}{S_{tt}}\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right).$$

$$\mathcal{E}(\hat{\beta}) = \frac{\mathcal{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i t_i - \bar{X}\bar{t}\right]}{S_{tt}}$$

$$= \frac{\frac{1}{n}\sum_{i=1}^{n} \mathcal{E}(X_i).t_i - \frac{1}{n}\sum_{i=1}^{n} \mathcal{E}(X_i)\bar{t}}{S_{tt}}$$

$$= \frac{\frac{1}{n}\sum_{i=1}^{n} (\alpha + \beta t_i) t_i - \frac{1}{n}\sum_{i=1}^{n} (\alpha + \beta t_i).\bar{t}}{S_{tt}}$$

$$= \frac{\cancel{\alpha \bar{t}} + \beta \frac{1}{n}\sum_{i=1}^{n} t_i^2 - \cancel{\alpha \bar{t}} - \beta \bar{t}^2}{S_{tt}}$$

$$= \frac{\beta\left(\frac{1}{n}\sum t_i^2 - \bar{t}^2\right)}{S_{tt}} = \frac{\beta . S_{tt}}{S_{tt}} = \beta.$$

$$\mathcal{E}(\hat{\alpha}) = \frac{1}{n}\sum_{i=1}^{n}\mathcal{E}(x_i) - \mathcal{E}(\hat{\beta}).\bar{t}$$

$$= \frac{1}{n}\sum_{i=1}^{n}(\alpha+\beta t_i) - \beta\bar{t}$$

$$= \alpha + \cancel{\beta\bar{t}} - \cancel{\beta\bar{t}}$$

$$= \alpha$$

$\therefore \hat{\alpha}$ & $\hat{\beta}$ are unbiased estimators. Hence, if we can show that they are functions of the complete & sufficient statistic, they will be UMVUE.

$$f(x_1,\ldots,x_n) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n}\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}(x_i-\alpha-\beta t_i)^2\right].$$

$$= \prod_{i=1}^{n}\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}(x_i^2+\alpha^2+\beta^2 t_i^2-2\beta x_i t_i-2\alpha x_i+2\alpha\beta t_i)\right]$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left[-\frac{\sum_{i=1}^{n}x_i^2}{2\sigma^2}+\frac{\beta}{\sigma^2}\sum_{i=1}^{n}x_i t_i+\frac{\alpha}{\sigma^2}\sum_{i=1}^{n}x_i-\frac{\beta^2}{2\sigma^2}\sum_{i=1}^{n}t_i^2 -\frac{\alpha\beta}{\sigma^2}\sum_{i=1}^{n}t_i-\frac{n\alpha^2}{2\sigma^2}\right].$$

$$= h(\underset{\sim}{x})\,c(\underset{\sim}{\theta})\exp\left[\sum_{i=1}^{3}\eta_i T_i(\underset{\sim}{x})\right]$$

where $\underset{\sim}{\eta} = \left(-\frac{1}{2\sigma^2}, \frac{\beta}{\sigma^2}, \frac{\alpha}{\sigma^2}\right)$ & $\underset{\sim}{T}(\underset{\sim}{x}) = \left(\sum_{i=1}^{n}x_i^2, \sum_{i=1}^{n}x_i t_i, \sum_{i=1}^{n}x_i\right)$.

$\therefore \underset{\sim}{T}(\underset{\sim}{x})$ is the required complete sufficient statistic for the exponential family $f(\underset{\sim}{x})$.

Now, $$\hat{\beta} = \frac{\frac{1}{n}\sum_{i=1}^{n}x_i t_i - \left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\bar{t}}{S_{tt}} = g_1(\underset{\sim}{T}(\underset{\sim}{x}))$$

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n}x_i - \hat{\beta}\bar{t} = g_2(\underset{\sim}{T}(\underset{\sim}{x})).$$

$\therefore \hat{\alpha}$ & $\hat{\beta}$ are functions of the complete sufficient statistic & they are unbiased $\Rightarrow \hat{\alpha}$ & $\hat{\beta}$ are the required UMVUEs.

Question 3 (Q3 of page 104).

$X \sim \text{Poisson}(\theta)$ ie $P(X=x) = e^{-\theta} \cdot \frac{\theta^x}{x!}$

Let us assume that $T(x)$ is an unbiased estimate of $1/\theta$.

$\therefore\ E(T(x)) = 1/\theta \quad \forall \theta$

$$\Leftrightarrow \sum_{x=0}^{\infty} e^{-\theta} \cdot \frac{\theta^x}{x!} \cdot T(x) = \frac{1}{\theta}$$

$$\Leftrightarrow \sum_{x=0}^{\infty} \frac{\theta^{x+1}\ T(x)}{x!} = e^{\theta} = \sum_{x=0}^{\infty} \frac{\theta^x}{x!}$$

Now since this result is true $\forall \theta$, the coefficients of θ^k must match on both sides. $\forall k$.

But the coefficient of θ^0 is on the LHS is 0, on the RHS is 1.

This is a contradiction.

Alternatively: $\rightarrow$ $\lim_{\theta \to 0} LHS = 0$, but $\lim_{\theta \to 0} RHS = 1$

$\Rightarrow$ $LHS \neq RHS$ in the limit.

So, $$\sum_{x=0}^{\infty} \frac{\theta^{x+1}}{x!} T(x) \neq \sum_{x=0}^{\infty} \frac{\theta^x}{x!}$$ for any $T(x)$.

$\Rightarrow$ $E(T(x)) \neq 1/\theta$ for any function of x.

So, $1/\theta$ cannot have an unbiased estimator.

[Proved].

Question 4 (Q5 of page 105)

For the functional model, $f_{\theta,\eta_i}(x,y) = \eta_i e^{-\eta_i x_i}\, \eta_i \theta\, e^{-\eta_i \theta y_i}\, I(x_i > 0, y_i > 0)$

$\therefore$ for this model, $\underset{\sim}{\theta}' = (\theta, \eta_1, \ldots, \eta_n)$

We want the information bound for $q(\underset{\sim}{\theta}) = (\theta, 0, \ldots, 0)'$.

$\therefore \nabla q(\theta) = (1, 0, \ldots, 0)'$.

$\therefore \nabla q(\underset{\sim}{\theta})' I^{-1}(\underset{\sim}{\theta}) \nabla q(\underset{\sim}{\theta})$ is basically the $(1,1)^{th}$ element of $I^{-1}(\underset{\sim}{\theta})$

Now, $l_{\underset{\sim}{\theta}} = \log\left[P_{\underset{\sim}{\theta}}(\underset{\sim}{x}, \underset{\sim}{y})\right] = 2\sum_{i=1}^{n} \log \eta_i + n \log \theta - \sum_{i=1}^{n} \eta_i (x_i + \theta y_i)$.

$$\therefore \frac{\partial l_{\underset{\sim}{\theta}}}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \eta_i y_i$$

$$\frac{\partial l_{\underset{\sim}{\theta}}}{\partial \eta_i} = \frac{2}{\eta_i} - (x_i + \theta y_i).$$

$$\therefore \frac{\partial^2 l_{\underset{\sim}{\theta}}}{\partial \theta^2} = -\frac{n}{\theta^2}, \quad \frac{\partial^2 l_{\underset{\sim}{\theta}}}{\partial \eta_i^2} = -\frac{2}{\eta_i^2}. \quad \frac{\partial^2 l_{\underset{\sim}{\theta}}}{\partial \eta_i \eta_j} = 0$$

$$\frac{\partial^2 l_{\underset{\sim}{\theta}}}{\partial \theta \partial \eta_i} = -y_i$$

$$\therefore I(\theta) = \begin{pmatrix} n/\theta^2 & \frac{1}{\eta_1\theta} & \frac{1}{\eta_2\theta} & \cdots & \frac{1}{\eta_n\theta} \\ \frac{1}{\eta_1\theta} & & & & \\ \vdots & & D_{n\times n} & & \\ \frac{1}{\eta_n\theta} & & & & \end{pmatrix}$$

where D is $\mathrm{diag}\left(\frac{2}{\eta_1^2}, \ldots, \frac{2}{\eta_n^2}\right)$.

Let $A = I(\underset{\sim}{\theta})^{-1}$ & $(t, t_1, \ldots, t_n)'$ be the 1st column of A.

Now, $I(\underset{\sim}{\theta}).A = I \Rightarrow \dfrac{nt}{\theta^2} + \sum_{i=1}^{n} \dfrac{t_i}{\eta_i \theta} = 1$.

Also, $\dfrac{t}{\eta_i \theta} + \dfrac{2t_i}{\eta_i^2} = 0 \Rightarrow \dfrac{t_i}{\eta_i} = -\dfrac{t}{2\theta}$.

$$\therefore \frac{nt}{\theta^2} - \sum_{i=1}^{n} \frac{t}{2\theta^2} = 1 \iff \frac{t}{\theta^2}(n - n/2) = 1 \text{ or } t = \frac{2\theta^2}{n}.$$

$\therefore$ the required information bound is $\frac{2\theta^2}{n} = \cancel{I(\theta)\ (say)}\ A_1$ (say).

(b) For the structural model,

$$P_{\theta,a,b}(x,y) = \int_0^\infty \theta\eta^2 e^{-\eta x} e^{-\eta\theta y} \frac{b^a \eta^{a-1}}{\Gamma_a} e^{-b\eta} d\eta$$

$$= \frac{\theta b^a}{\Gamma_a} \int_0^\infty \eta^{a+1} e^{-\eta(x+\theta y+b)} d\eta.$$

$$= \frac{\theta b^a}{\Gamma_a} \cdot \frac{\Gamma_{a+2}}{(x+\theta y+b)^{a+2}} \quad \text{[from integral of Gamma density]}$$

$$= \frac{a(a+1)\cdot\theta b^a}{(x+\theta y+b)^{a+2}}.$$

$$\therefore \log[P_{\theta,a,b}(x,y)] = k + \log\theta - (a+2)\log(x+\theta y+b). \quad \text{[k indep of } \theta]$$

$$\therefore l_\theta = \log[P_{\theta,a,b}(\underset{\sim}{x},\underset{\sim}{y})] = c + n\log\theta - (a+2)\sum_{i=1}^n \log(x_i+\theta y_i+b).$$

[c indep of θ]

$$\therefore \frac{\partial l_\theta}{\partial\theta} = \frac{n}{\theta} - (a+2)\sum_{i=1}^n \frac{y_i}{x_i+\theta y_i+b}$$

$$\therefore \frac{\partial^2 l_\theta}{\partial\theta^2} = -\frac{n}{\theta^2} + (a+2)\sum_{i=1}^n \frac{y_i^2}{(x_i+\theta y_i+b)^2}$$

$$\therefore I(\theta) = \mathcal{E}\left[-\frac{\partial^2 l_\theta}{\partial\theta^2}\right]$$

$$= \frac{n}{\theta^2} - \mathcal{E}\left[(a+2)\sum_{i=1}^n \frac{y_i^2}{(x_i+\theta y_i+b)^2}\right].$$

$$= \frac{n}{\theta^2} - n(a+2).\ \mathcal{E}\left[\frac{y_1^2}{(x_1+\theta y_1+b)^2}\right] \quad [\because (x_i,y_i)\text{'s are iid}].$$

$$\mathcal{E}\left[\frac{y_1^2}{(x_1+\theta y_1+b)^2}\right]$$

$$= \int_0^\infty\int_0^\infty \frac{y_1^2}{(x_1+\theta y_1+b)^2}\cdot\frac{a(a+1)\,\theta b^a}{(x_1+\theta y_1+b)^{a+2}}\,dx_1 dy_1$$

$$= \int_0^\infty y_1^2 \int_{\theta y_1+b}^\infty \frac{a(a+1)\,\theta b^a}{z^{a+4}}\,dz\,dy_1 \qquad [z = x_1+\theta y_1+b].$$

$$= \int_0^\infty y_1^2 \;\frac{a(a+1)\,.\theta b^a\,(\theta y_1+b)^{-(a+3)}}{(a+3)}\,dy_1$$

$$= \frac{a(a+1)}{a+3}\int_b^\infty\left(\frac{\omega-b}{\theta}\right)^2\cdot\,\theta b^a\,\omega^{-(a+3)}\,d\omega/\theta \qquad [\text{Assuming } \omega = \theta y_1+b].$$

$$= \frac{a(a+1)\,b^a}{(a+3)\,\theta^2}\left[\int_b^\infty \omega^{-(a+1)}d\omega - 2b\int_b^\infty \omega^{-(a+2)}d\omega + b^2\int_b^\infty \omega^{-(a+3)}d\omega\right]$$

$$= \frac{a(a+1)\,b^a}{(a+3)\,\theta^2}\cdot\left[\frac{1}{ab^a} - \frac{2b}{(a+1)b^{a+1}} + \frac{b^2}{(a+2)b^{a+2}}\right].$$

$$= \frac{a(a+1)\cancel{b^a}}{(a+3)\,\theta^2}\cdot\frac{1}{\cancel{b^a}}\left[\frac{1}{a} - \frac{2}{a+1} + \frac{1}{a+2}\right].$$

$$= \frac{a(a+1)}{(a+3)\theta^2}\left[\frac{\cancel{a}+1-\cancel{a}}{a(a+1)} - \frac{\cancel{a}+2-\cancel{a}-1}{(a+1)(a+2)}\right]$$

$$= \frac{a(a+1)}{(a+3)\theta^2}\left[\frac{1}{a(a+1)} - \frac{1}{(a+1)(a+2)}\right]$$

$$= \frac{a(a+1)}{(a+3)\,\theta^2}\cdot\frac{2}{a(a+1)(a+2)} = \frac{2}{(a+3)(a+2)\theta^2}.$$

$\therefore\ I(\theta) = \dfrac{n}{\theta^2} - \dfrac{\cancel{(a+2)}n}{(a+3)\cancel{(a+2)}} \cdot \dfrac{2}{\theta^2}$

$= \dfrac{n}{\theta^2}\left[1 - \dfrac{2}{a+3}\right] = \dfrac{n}{\theta^2}\left(\dfrac{a+1}{a+3}\right).$

$\therefore$ the required information bound is $I(\theta)^{-1} = \dfrac{\theta^2}{n}\dfrac{(a+3)}{(a+1)} = A_2$ (say)

(c) Hence, $A_1 \gtreqless A_2$ according as

$$\frac{2\theta^2}{n} \gtreqless \frac{\theta^2}{n}\frac{(a+3)}{a+1}$$

$$\Leftrightarrow \quad a \gtreqless 1.$$

So, if $a>1$, the information bound in (a) is greater,
if $a<1$, the information bound in (b) is greater,
if $a=1$, they are equal.

Now, the information for θ in (a) is $\dfrac{n}{\theta^2}$ & in (b) is

$\dfrac{n}{\theta^2} \cdot \dfrac{a+1}{a+3}$.

$\Rightarrow$ information for θ in (a) is always greater than that in (b)

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