1. We identify the kernel function $h(X_1, X_2) = |X_1 - X_2|$. Apply the CLT for the U-statistics and after the calculation under the uniform distribution, we obtain

$$\sqrt{n}\left\{\binom{n}{2}^{-1}\sum\sum_{i<j}|X_i - X_j| - \frac{1}{3}\right\} \to N(0, \frac{1}{90}).$$

2. Clearly, $E[|X_n|] < \infty$. Let $\mathcal{F}_n = \sigma(Y_1, ..., Y_n)$. Then $\mathcal{F}_1 \subset \mathcal{F}_2 \subset ....$ Since

$$X_{n+1} = (\sum_{k=1}^{n} Y_k)^2 + 2(\sum_{k=1}^{n} Y_k)Y_{n+1} + Y_{n+1}^2 - (n+1)\sigma^2,$$

we have

$$E[X_{n+1}|\mathcal{F}_n] = (\sum_{k=1}^{n} Y_k)^2 + 2(\sum_{k=1}^{n} Y_k)E[Y_{n+1}|Y_1, ..., Y_n] + E[Y_{n+1}^2|Y_1, ..., Y_n] - (n+1)\sigma^2$$

$$= (\sum_{k=1}^{n} Y_k)^2 + 2(\sum_{k=1}^{n} Y_k)E[Y_{n+1}] + E[Y_{n+1}^2] - (n+1)\sigma^2 = (\sum_{k=1}^{n} Y_k)^2 - n\sigma^2 = X_n.$$

Therefore, $\{X_n\}$ is a martingale.

3. (a) Clearly, $E[T_n] = \sum_{i=1}^{n} \omega_{ni}\mu = \mu$ and

$$Var(T_n) = \sum_{i=1}^{n} \omega_{ni}^2 \sigma_i^2.$$

We aim to minimize $Var(T_n)$ understand constraint $\sum_{i=1}^{n} \omega_{ni} = 1$ and $\omega_{ni} \geq 0$. This can be done using the Lagrange multiplier by solving

$$\sigma_i^2 \omega_{ni} - \lambda = 0.$$

This yields $\omega_{ni} = 1/(\lambda\sigma_i^2)$. Since $\sum_{i=1}^{n} \omega_{ni} = 1$, we obtain the optimal weight.

(b) When $\omega = \omega^{opt}$,

$$Var(T) = \left\{\sum_{i=1}^{n} 1/\sigma_i^2\right\}^{-1}.$$

Since $Var(T) \to 0$, by the Chebyshev's inequality,

$$P(|T_n - \mu| > \epsilon) \leq \epsilon^{-2} Var(T_n) \to 0.$$

Thus, $T_n \to_p \mu$.

(c) When $X_i = \mu + \sigma_i \epsilon_i$,

$$\sqrt{\sum_{i=1}^{n}(1/\sigma_i^2)}(T_n - \mu) = \sum_{i=1}^{n} \frac{1/\sigma_i}{\sqrt{\sum_{i=1}^{n}(1/\sigma_i^2)}}\epsilon_i.$$

We verify the Lindeberg conditions. Clearly, the above variance is 1. Furthermore, for any $\delta > 0$,

$$\sum_{i=1}^{n} E\left[\left(\frac{1/\sigma_i}{\sqrt{\sum_{i=1}^{n}(1/\sigma_i^2)}}\epsilon_i\right)^2 I\left(\frac{1/\sigma_i}{\sqrt{\sum_{i=1}^{n}(1/\sigma_i^2)}}|\epsilon_i| > \delta\right)\right]$$

$$\leq \sum_{i=1}^{n} \frac{1/\sigma_i^2}{\sum_{i=1}^{n}(1/\sigma_i^2)} E\left[\epsilon_1^2 I\left(|\epsilon_1|^2 > \delta^2/[\max(1/\sigma_i^2)/\sum_{j=1}^{n}(1/\sigma_j^2)]\right)\right]$$

$$\leq E\left[\epsilon_1^2 I\left(|\epsilon_1|^2 > \delta^2/[\max(1/\sigma_i^2)/\sum_{j=1}^{n}(1/\sigma_j^2)]\right)\right] \to 0.$$

Hence, the result follows from the Lindeberg-Feller CLT.

(d)

$$Var(T_n)/Var(\bar{X}_n) = \frac{n^2}{\sum_{j=1}^{n}(1/\sigma_j^2)\sum_{j=1}^{n}\sigma_j^2}.$$

The following table gives the ratios for difference choices of r and n:

r	$n = 5$	$n = 10$	$n = 20$	$n = 50$	$n = 100$	$n = \infty$
0.25	5.503909e-02	2.145771e-04	8.185452e-10	4.437343e-27	1.400178e-56	0
0.5	4.162331e-01	4.892363e-02	1.907352e-04	1.110223e-12	3.944305e-27	0
0.75	8.498904e-01	5.269579e-01	1.063807e-01	1.179838e-04	2.672668e-10	0