BIOS760 HOMEWORK V SOLUTION

1. (a) S_n has the Poisson distribution with parameter n. Thus,

$$E[\left\{(S_n - n)/\sqrt{n}\right\}^-] = \sum_{k=0}^{n} \frac{n-k}{\sqrt{n}} \frac{n^k e^{-n}}{k!} = \sqrt{n}e^{-n}\left\{\sum_{k=0}^{n} \frac{n^k}{k!} - \sum_{k=1}^{n} \frac{n^{k-1}}{(k-1)!}\right\} = \sqrt{n}e^{-n}\frac{n^n}{n!}.$$

(b) By the CLT, $(S_n - n)/\sqrt{n} \to_d Z$. Since $g(x) = \max(-x, 0)$ is continuous, by continuous mapping theorem

$$\left\{(S_n - n)/\sqrt{n}\right\}^- \to_d Z^-.$$

(c) Since

$$E[|\left\{(S_n - n)/\sqrt{n}\right\}^-|^2] \le E[(S_n - n)^2/n] = 1,$$

$\{(S_n - n)/\sqrt{n}\}^-$ satisfies the uniform integrability condition from the Liaponov condition. Note that the Vitali's theorem also holds for $X_n \to_d X$ (use the Skorohod's representation). We obtain the result.

(d) Note $P(Z^- \le x) = 0$ if $x < 0$ and $1 - \Phi(-x)$ if $x \ge 0$. Then $E[Z^-] = 0 \times 1/2 - \int_{-\infty}^{0} x\phi(x)dx = 1/\sqrt{2\pi}$. Thus, (a) and (b) imply that

$$\sqrt{n}e^{-n}\frac{n^n}{n!} \to 1/\sqrt{2\pi},$$

i.e., $n! \sim \sqrt{2\pi}n^{n+1/2}e^{-n}$.

2. (a)

$$\begin{aligned} & |E[e^{i(t_1X_n+t_2Y_n)}] - E[e^{i(t_1X+t_2y)}]| \\ \le\ & |E[e^{it_1X_n}(e^{it_2Y_n} - e^{it_2y})]| + |E[e^{it_2y}e^{it_1X_n}] - E[e^{it_2y}e^{it_1X}]| \\ \le\ & E[|e^{it_2Y_n} - e^{it_2y}|] + |E[e^{it_1X_n}] - E[e^{it_1X}]|. \end{aligned}$$

Since $Y_n \to_p y$, $|e^{it_2Y_n} - e^{it_2y}| \to_p 0$. By the DCT, the first term converges to zero. The second term vanishes as $n \to \infty$ since $X_n \to_d X$. We obtain that the characteristic function of $(X_n, Y_n)'$ converges to the characteristic function of $(X, y)'$. Thus $(X_n, Y_n)' \to_d (X, y)'$.

(b) Since $X_n \to_d X$ and $Z_n \to_d z$, from (a), $(X_n, Z_n)' \to_d (X_n, z)'$. $g(x, z) = xz$ is a continuous function on R^2. By the continuous mapping theorem,

$$Z_n X_n \to_d zX.$$

Moreover, since $Y_n \to_p y$, $(Z_n X_n, Y_n)' \to_d (zX, y)'$. Since $g(x, y) = x + y$ is continuous, we obtain

$$Z_n X_n + Y_n \to_d zX + y.$$

3. For any open set G, it suffices to show that

$$\liminf_n P(X_n \in G) \geq P(X \in G).$$

For any constant M, consider $O = G \cap (-M, M)$. As in proving (c) of the Portmanteau Theorem, we construct a function

$$g(x) = 1 - \frac{\epsilon}{\epsilon + d(x, O^c)}.$$

Then $g(x)$ is bounded continuous and $0 \leq g(x) \leq 1$. Additionally, $g(x) = 0$ when $x \in O^c$. Since O^c contains the complement of $(-M, M)$, g has a bounded support. Thus, from the condition, we obtain $E[g(X_n)] \to E[g(X)]$. As a result,

$$\liminf_n P(X_n \in G) \geq \liminf_n P(X_n \in O)$$

$$\geq \liminf_n E[g(X_n)] = E[g(X)] = E[1 - \frac{\epsilon}{\epsilon + d(X, O^c)}].$$

Let ϵ decrease to zero. Note the right-hand side increases to $E[I(X \in O)]$. We have

$$\liminf_n P(X_n \in G) \geq P(X \in G \cap (-M, M)).$$

Let $M \to \infty$ then $\liminf_n P(X_n \in G) \geq P(X \in G)$.

4. (a) $P(M_n - \alpha^{-1}\log n \leq x) = P(X_1 \leq x + \alpha^{-1}\log n)^n = (1 - \exp\{-\alpha x - \log n\})^n I(x > -\alpha^{-1}\log n) \to \exp\{-e^{-\alpha x}\}$.

(b) $P(n^{-1/\alpha}M_n \leq x) = G(n^{1/\alpha}x)^n$. When $x \leq 0$, $n^{1/\alpha}x \leq 1$, the probability is zero. When $x > 0$, $n^{1/\alpha}x \geq 1$ eventually. Then the probability is equal to $(1 - n^{-1}x^{-\alpha})^n \to \exp\{-x^{-\alpha}\}$. Thus,

$$P(n^{-1/\alpha}M_n \leq x) \to \exp\{-x^{-\alpha}\}I(x > 0).$$

(c) When n is large,

$$P(n^{1/\alpha}(M_n - 1) \leq x) = G(1 + n^{-1/\alpha}x)^n = \begin{cases} 1 & \text{if } x \geq 0, \\ (1 - n^{-1}(-x)^\alpha)^n & \text{if } x < 0. \end{cases}$$

Thus,

$$P(n^{1/\alpha}(M_n - 1) \leq x) \to \exp\{-(-x)^\alpha\} I(x < 0) + I(x \geq 0).$$