BIOS760 HOMEWORK IV SOLUTION

1. If we can show

$$\int_\Omega g^+(X(\omega))dP(\omega) = \int_F g^+(x)dP_X(x)$$

and

$$\int_\Omega g^-(X(\omega))dP(\omega) = \int_F g^-(x)dP_X(x),$$

then the result holds due to the fact that $g(X)$ is integrable. Thus, it is sufficient to prove the result assuming $g \geq 0$. We first prove that the equality holds when g is a simple function: assume $g(x) = \sum_{i=1}^n x_i I(x \in B_i)$, where $B_1, ..., B_n$ are disjoint Borel sets in $\mathcal{B}$. Then

$$\int_\Omega g(X(\omega))dP(\omega) = \sum_{i=1}^n x_i \int_\Omega I(\{\omega : X(\omega) \in B_i\})dP(\omega)$$

$$= \sum_{i=1}^n x_i P(X^{-1}(B_i)) = \sum_{i=1}^n x_i P_X(B_i) = \int_R g(x)dP_X(x).$$

For any non-negative measurable function g, we choose a sequence of simple functions, $0 \leq g_n \leq g$, and g_n increasing to g. Then

$$\int_\Omega g(X(\omega))dP(\omega) = \lim_n \int_\Omega g_n(X(\omega))dP(\omega) = \lim_n \int_R g_n(x)dP_X(x) = \int_R g(x)dP_X(x).$$

2. (a) $F(x,y) = P(X \leq x, Y \leq y) = P(X \leq \min(x, y/2)) = \Phi(\min(x, y/2))$ so it is continuous.

   (b) Since $\lambda \times \lambda(\{(x,y) : y = 2x\}) = 0$ but $P_{(X,Y)}(\{(x,y) : y = 2x\}) = 1$, $P_{(X,Y)}$, equivalently, $F(x,y)$, is not absolutely continuous with respect to the Lebesgue measure.

3. (a) We first show the inequality in the hint:

$$\int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx \leq \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-x^2(1-\delta)/2} e^{-x^2\delta/2}dx$$

$$\leq \frac{1}{\sqrt{2\pi}} e^{-y^2(1-\delta)/2} \int_{-\infty}^\infty e^{-x^2\delta/2}dx = e^{-y^2(1-\delta)/2}/\sqrt{\delta},$$

$$\int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx \geq \int_y^{(1+\delta)y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx \geq \frac{\delta}{\sqrt{2\pi}} e^{-(1+\delta)y^2/2} y.$$

For any $\epsilon > 0$, we can choose δ small enough so that

$$(1+\epsilon)^2(1-\delta) > 1, \quad (1-\epsilon)^2(1+\delta) < 1.$$

Then

$$P(N(0,1) > \sqrt{2\log n}(1+\epsilon)) \le \frac{1}{\sqrt{\delta}} \exp\{-2\log n(1+\epsilon)^2(1-\delta)/2\} = \frac{1}{\sqrt{\delta}} n^{-(1+\epsilon)^2(1-\delta)}$$

so

$$\{P(N(0,1) \le \sqrt{2\log n}(1+\epsilon))\}^n \ge (1 - n^{-(1+\epsilon)^2(1-\delta)}/\sqrt{\delta})^n \to 1.$$

On the other hand,

$$P(N(0,1) < \sqrt{2\log n}(1-\epsilon)) = 1 - P(N(0,1) \ge \sqrt{2\log n}(1-\epsilon))$$

$$\le 1 - \frac{\delta}{\sqrt{2\pi}} e^{-(1+\delta)\log n(1-\epsilon)^2} \sqrt{2\log n}(1-\epsilon) = 1 - \frac{\delta(1-\epsilon)}{\sqrt{\pi}} \sqrt{\log n} n^{-(1+\delta)(1-\epsilon)^2}$$

so

$$\{P(N(0,1) < \sqrt{2\log n}(1-\epsilon))\}^n \le \{1 - \frac{\delta(1-\epsilon)}{\sqrt{\pi}} \sqrt{\log n} n^{-(1+\delta)(1-\epsilon)^2}\}^n \to 0.$$

Therefore,

$$P(|X_{(n)}/\sqrt{2\log n} - 1| > \epsilon) = P(X_{(n)} > \sqrt{2\log n}(1+\epsilon)) + P(X_{(n)} < \sqrt{2\log n}(1-\epsilon))$$

$$= 1 - (P(N(0,1) \le \sqrt{2\log n}(1+\epsilon)))^n + P(N(0,1) < \sqrt{2\log n}(1-\epsilon))^n \to 0.$$

That is, $X_{(n)}/\sqrt{2\log n} \to_p 1$.

(b) For any $x \le 0$, $P(n(1-X_{(n)}) \le x) = 0$. For any $x > 0$,

$$P(n(1-X_{(n)}) \le x) = P(X_{(n)} \ge (1-x/n)) = 1 - P(X_{(n)} < 1 - x/n)$$

$$= 1 - (1-x/n)^n \to 1 - e^{-x}.$$

Thus, $n(1-X_{(n)}) \to_d Exp(1)$.

4. (a) $X_n \to_{a.s} 0$ is clear since X_n's support shrinks to zero. $E[X_n] = 1/\log(n+1) \to 0$.

(b) We can not. If $Y \ge |X_n|$, then Y is at least $n^\alpha/\log(n+1)$ when U is in $((n+1)^{-\alpha}, n^{-\alpha}]$. Thus,

$$E[Y] \ge \sum_{n=1}^{\infty} \frac{n^\alpha}{\log(n+1)} E[I(U \in ((n+1)^{-\alpha}, n^\alpha])] \ge \sum_{n=1}^{\infty} \frac{n^\alpha}{\log(n+1)} \left\{ \frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha} \right\}$$

$$\ge \sum_{n=1}^{\infty} \frac{(n+1)^\alpha/2^\alpha}{2(\log n + 1)} \left\{ \frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha} \right\}$$

$$\ge \sum_{n=1}^{\infty} \int_{(n+1)^{-\alpha}}^{n^{-\alpha}} \frac{1/(2^\alpha x)}{2(\log x^{-1/\alpha} + 1)} dx = \frac{\alpha}{2^{\alpha+1}} \int_0^1 \frac{1}{-x(\log x - \alpha)} dx = \infty.$$

(c) For fixed M, when n is large enough, $|X_n| I_{|X_n| \geq M} = X_n$. Thus, for any $\alpha > 0$, the uniform integrability condition holds.

5. (a) $f_n(x) = 1 + \cos 2\pi n x$ does not converge at any $x \in (0, 1)$. However, the corresponding CDF $F_n(x) = x + \frac{\sin 2\pi n x}{2\pi n} \to x$.

   (b) $S_n = X_1 + ... + X_n$, where $X_1, .., X_n$ are i.i.d $N(0, 1)$. Thus, S_n/n has a continuous density but $S_n/n \to_d 0$.

   (c) $S_n = X_1 + ... + X_n$, where $X_1, .., X_n$ are i.i.d Bernoulli(p). S_n is discrete but

$$\sqrt{n}(S_n/n - p) \to_d N(0, 1).$$