BIOS760 HOMEWORK III SOLUTION

1. The result can be shown using two different ways.

Proof 1. Since $|X|I(|X| \le n)$ increases to $|X|$, by the monotone convergence theorem,

$$\lim_{n\to\infty} \int |X|I(|X| \le n)d\mu = \int |X|d\mu.$$

Since $\int |X|d\mu < \infty$,

$$\lim_{n\to\infty} \int |X|I(|X| > n)d\mu = 0.$$

We can choose a large n such that

$$\int |X|I(|X| > n)d\mu < \frac{\epsilon}{2}.$$

Note that for any A,

$$\begin{aligned}\int_A |X|d\mu &= \int_A |X|I(|X| \le n)d\mu + \int_A |X|I(|X| > n)d\mu \\ &\le n \int_A d\mu + \int |X|I(|X| > n)d\mu \le n\mu(A) + \frac{\epsilon}{2}.\end{aligned}$$

Thus, if $\mu(A) < \delta = \epsilon/2n$, then

$$\int_A |X|d\mu < \epsilon.$$

Proof 2. We prove by contradiction. If not, then there exists some $\epsilon > 0$ such that for any $\delta > 0$, we can always find some A such that $\mu(A) < \delta$ but $\int_A |X|d\mu \ge \epsilon$. Particularly, we choose $\delta = 1/2^n$ and obtain A_n such that $\mu(A_n) < 1/2^n$ and $\int_{A_n} |X|d\mu \ge \epsilon$. Let $A = \limsup_n A_n = \cap_{n=1}^\infty \cup_{m\ge n} A_m$. Then

$$\mu(A) \le \mu(\cup_{m\ge n} A_m) \le \sum_{m=n}^\infty \mu(A_m) = \sum_{m=n}^\infty \frac{1}{2^m} \to 0, \quad n \to \infty.$$

Thus, $\mu(A) = 0$. However, since $I_{\cup_{m\ge n} A_m}|X|$ converges to $I_A|X|$ and they are bounded by $|X|$, by the dominant convergence theorem,

$$\lim_{n\to\infty} \int_{\cup_{m\ge n} A_m} |X|d\mu = \int_A |X|d\mu = 0.$$

However, for each n,

$$\int_{\cup_{m\ge n} A_m} |X|d\mu \ge \int_{A_n} |X|d\mu \ge \epsilon.$$

We have the contradiction.

2. Let B be the set of all rational numbers then $\nu(B) > 0$ but $\mu(B) = 0$. Thus, ν is not dominated by μ. Let $B = \{x : x > 1 \text{ and } x \text{ is irrational}\}$ then $\nu(B) = 0$ but $\mu(B) > 0$. Thus, μ is not dominated by ν. However, since whenever $(\mu+\nu)(B) = \mu(B)+\nu(B) = 0$, it implies $\nu(B) = 0$, $\nu \prec\prec (\mu+\nu)$. To calculate the Radon-Nikodym derivative of ν with respect to $(\mu + \nu)$, it is equivalent to find a non-negative and measurable function $f(x)$ such that for any Borel set B,

$$\nu(B) = \int_B f(x) d(\mu + \nu)(x).$$

Explicitly write out the above equation using the definition of ν. We obtain

$$\int_B I(x \in [0,1]) d\mu(x) + \sum_{r_i \in B} 2^{-i} = \int_B f(x) d\mu + \int_B f(x) I(x \in [0,1]) d\mu(x) + \sum_{r_i \in B} f(r_i) 2^{-i}.$$

That is,

$$\int_B \{I(x \in [0,1]) - (1 + I(x \in [0,1])) f(x)\} \, d\mu(x) + \sum_{r_i \in B} (1 - f(r_i)) 2^{-i} = 0.$$

First, choose $B = \{r_i\}$ and we obtain

$$(1 - f(r_i)) 2^{-i} = 0 \quad \text{so} \quad f(r_i) = 1.$$

Therefore, the forgoing equation becomes

$$\int_B f(x) d\mu + \int_B f(x) I(x \in [0,1]) d\mu(x) = \int_{B \cap Q^c} f(x)(1 + I(x \in [0,1])) d\mu(x) = 0,$$

where Q denotes the set of all the rational numbers. Thus, we can let $f(x)(1 + I(x \in [0,1])) = 0$ for $x \notin Q$. That is, $f(x)$ is given as

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ I(x \in [0,1])/(1 + I(x \in [0,1])) = I(x \in [0,1])/2, & x \text{ irrational} \end{cases}$$

It is easy to check this function satisfies the above equation so by the uniqueness, $f(x)$ is the derivative.

3. Since $E[X_1|X_{(n)}]$ is a measurable function of $X_{(n)}$, we denote it as $g(X_{(n)})$. For any $x \in (0,1)$, in the following equality

$$E[I(X_{(n)} \le x) g(X_{(n)})] = E[I(X_{(n)} \le x) X_1],$$

we obtain that the right-hand side is equal to

$$E[X_1 I(X_1 \le x) I(X_2 \le x, ..., X_n \le x)] = E[X_1 I(X_1 \le x)] P(X_2 \le x)^{n-1} = x^{n+1}/2.$$

However, since $X_{(n)}$ has a density function $nx^{n-1}I(0 \le x \le 1)$, the left-hand side of the equation should be equal to $\int_0^x g(z)nz^{n-1}dz$. We have

$$\int_0^x g(z)nz^{n-1}dz = x^{n+1}/2.$$

Differentiate both sides with respect to x then $g(x) = (n+1)x/2n$. That is,

$$E[X_1|X_{(n)}] = \frac{n+1}{2n}X_{(n)}.$$

4. We first prove " $\Rightarrow$". Since in $A \cap B^c$, $g(y) = 0$, $P_Y(A \cap B^c) = \int_{A \cap B^c} g(y)d\lambda(y) = 0$. By the absolutely continuity, $P_X(A \cap B^c) = 0$. Since

$$A \cap B^c = \cup_{k=1}^{\infty} \left[\{x : f(x) > 1/k\} \cap B^c\right],$$

for any k,

$$0 = P_X(A \cap B^c) \ge P_X(\{x : f(x) > 1/k\} \cap B^c) = \int_{\{x:f(x)>1/k\} \cap B^c} f(x)d\lambda(x)$$

$$\ge \frac{1}{k}\lambda(\{x : f(x) > 1/k\} \cap B^c).$$

We obtain

$$\lambda(\{x : f(x) > 1/k\} \cap B^c) = 0.$$

Thus, $\lambda(A \cap B^c) = 0$.

For the direction " $\Leftarrow$", suppose $P_Y(C) = 0$ for some Borel set C. Using the similar argument as above, we can show $\lambda(C \cap B) = 0$. Since $\lambda(A \cap B^c) = 0$, we obtain $\lambda(C \cap A) = 0$. Therefore,

$$P_X(C) = \int_C f(x)d\lambda(x) = \int_{A^c \cap C} f(x)d\lambda(x) + \int_{A \cap C} f(x)d\lambda(x) = 0.$$

That is, $P_X \prec\prec P_Y$.

The last statement is true since $\{x : \phi(x) > 0\} = R$ and $\left\{y : I_{[0,1]}(y) > 0\right\} = [0, 1]$.