1. For any open set $O \in \mathcal{O}$, we claim that

$$O = \text{union of all open intervals } (x - r, x + r)$$

such that $x$ and $r$ are rational numbers and $(x - r, x + r) \subset O$.

Obviously, the right-hand side is contained in $O$. For any $y \in O$, we can find a positive number $\epsilon$ such that $(y - \epsilon, y + \epsilon) \subset O$, we choose $x$ to be a rational number such that $|x - y| < r < \epsilon/2$ where $r$ is another rational number. Then $y \in (x - r, x + r) \subset (y - \epsilon, y + \epsilon) \subset O$. We have proved the claim. Since the union is countable and $(x - r, x + r) = \bigcup_{n=1}^{\infty} (x - r, x + r - r/n) \in \mathcal{B}$, $O \in \mathcal{B}$. Thus $\mathcal{O} \subset \mathcal{B}$ then $\sigma(\mathcal{O}) \subset \mathcal{B}$. On the other hand, for any left-open and right-close interval $(a, b]$, $(a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n) \in \mathcal{O} \subset \sigma(\mathcal{O})$. We have $\mathcal{B} \subset \sigma(\mathcal{O})$. The result holds.

2. First, we show $\tilde{\mathcal{A}}$ is a $\sigma$-field. Clearly, $\emptyset$ and $\Omega$ belong to $\tilde{\mathcal{A}}$. If $\tilde{A} \in \tilde{\mathcal{A}}$, then there exists $A$ and $N$ such that $\tilde{A} = A \cup N$, $A \in \mathcal{A}$ and $N \subset B \in \mathcal{A}$ with $\mu(B) = 0$. Then $A^c = A^c \cap N^c$. Since $N^c = (B - (B - N))^c = B^c \cup (B - N)$, we have

$$\tilde{A}^c = (A^c \cap B^c) \cup (A^c \cap (B - N)).$$

Clearly, $A^c \cap B^c \in \mathcal{A}$ and $A^c \cap (B - N) \subset B$ with $\mu(B) = 0$. Thus, $\tilde{A}^c \in \tilde{\mathcal{A}}$. If $\tilde{A}_1, \tilde{A}_2, ... \in \tilde{\mathcal{A}}$, then there exists $A_1, A_2, ... \in \mathcal{A}$ and $N_1, N_2, ...$ such that $\tilde{A}_i = A_i \cup N_i$ and $N_i \subset B_i$ for some $B_i \in \mathcal{A}$ with $\mu(B_i) = 0$, $i = 1, 2, ...$. We obtain

$$\bigcup_{n=1}^{\infty} \tilde{A}_n = \bigcup_{n=1}^{\infty} \{A_i \cup N_i\} = (\bigcup_{n=1}^{\infty} A_i) \cup (\bigcup_{n=1}^{\infty} N_i).$$

Since $\bigcup_{n=1}^{\infty} A_i \in \mathcal{A}$ and $\bigcup_{n=1}^{\infty} N_i \subset \bigcup_{n=1}^{\infty} B_i \in \mathcal{A}$ with $\mu(\bigcup_{n=1}^{\infty} B_i) \leq \sum_{n=1}^{\infty} \mu(B_i) = 0$, we conclude $\bigcup_{n=1}^{\infty} A_i \in \tilde{\mathcal{A}}$. So $\tilde{\mathcal{A}}$ is a $\sigma$-field. Second, we show $\tilde{\mu}$ is a measure. If $\tilde{A}_1 = A_1 \cup N_1, \tilde{A}_2 = A_2 \cup N_2, ...$ are disjoint in $\tilde{\mathcal{A}}$, so are $A_1, A_2, ...$. From the previous argument, we have

$$\bigcup_{n=1}^{\infty} \tilde{A}_n = \bigcup_{n=1}^{\infty} \{A_i \cup N_i\} = (\bigcup_{n=1}^{\infty} A_i) \cup (\bigcup_{n=1}^{\infty} N_i),$$

where $\bigcup_{n=1}^{\infty} N_i$ is contained in a zero-measure set. Thus,

$$\tilde{\mu}(\bigcup_{n=1}^{\infty} \tilde{A}_n) = \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \tilde{\mu}(A_n).$$
3. (a) \( F(-\infty) = P(\emptyset) = 0 \) and \( F(\infty) = P(\mathbb{R}) = 1 \). For any sequence \( x_n \) decreasing to \( x \), the sequence of sets \( \{(-\infty, x_n]\} \) decreases to \( (-\infty, x] \). Moreover, \( P((-\infty, x_1]) < 1 \).

\[
\lim_{n} F(x_n) = \lim_{n} P((-\infty, x_n]) = P((-\infty, x]) = F(x).
\]

That is, \( F(x) \) is right-continuous.

(b) Both \( P \) and \( \mu_F \) are two measures on \( \mathcal{B} \). Moreover, \( P((a, b]) = P((-\infty, b]) - P((-\infty, a]) = F(b) - F(a) = \mu_F((a, b]) \). Therefore, both \( P \) and \( \mu_F \) can be treated as the extension of the same measure in \( \mathcal{B}_0 \) to \( \mathcal{B} \). By the uniqueness in the Caratheodory extension theorem, \( P(B) = \mu_F(B) \) for any \( B \in \mathcal{B} \).