Solution to Homework #10

Question 1 (Q4 of page 127)

(a) Using Taylor Expansion, we get

$$\sum_{i=1}^{n} \ddot{l}_{\hat{\underset{\sim}{\theta}}_n}(x_i) = \sum_{i=1}^{n} \ddot{l}_{\underset{\sim}{\theta}}(x_i) + \left\{ \sum_{i=1}^{n} \dddot{l}_{\tilde{\underset{\sim}{\theta}}_n}(x_i) \right\} (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})$$ where $\tilde{\theta}_n$ lies between $\underset{\sim}{\theta}$ & $\hat{\underset{\sim}{\theta}}_n$.

$$\left| \frac{1}{n} \sum_{i=1}^{n} \ddot{l}_{\hat{\underset{\sim}{\theta}}_n}(x_i) - \frac{1}{n} \sum_{i=1}^{n} \ddot{l}_{\underset{\sim}{\theta}}(x_i) \right| \le \left\{ \frac{1}{n} \sum_{i=1}^{n} \left| M_{\tilde{\underset{\sim}{\theta}}_n}(x_i) \right| + o_p(1) \right\} |\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}|$$

$$= \mathbb{P}_n \left[M_{\tilde{\theta}_n}(x) \right] |\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}| + o_p(1).$$

$$\xrightarrow{P} 0 \quad \left[\because \ |\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}| \xrightarrow{P} 0 \text{ as } \sqrt{n}(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}) \xrightarrow{d} N(0, I_\theta^{-1}) \right.$$
$$\left. \& \ \sup_\theta \mathcal{E}_\theta[M(x)] = M < \infty \Rightarrow \mathbb{P}[M_{\tilde{\underset{\sim}{\theta}}_n}(x)] \text{ is bounded} \right]$$

Now, by WLLN, $-\frac{1}{n} \sum_{i=1}^{n} \ddot{l}_\theta(x_i) \longrightarrow I_\theta$

$$\therefore \hat{I}_n = -\frac{1}{n} \sum_{i=1}^{n} \ddot{l}_{\hat{\underset{\sim}{\theta}}_n}(x_i)$$
$$= -\left[\frac{1}{n} \sum_{i=1}^{n} \ddot{l}_{\hat{\underset{\sim}{\theta}}_n}(x_i) - \frac{1}{n} \sum_{i=1}^{n} \ddot{l}_{\underset{\sim}{\theta}}(x_i) \right] - \frac{1}{n} \sum_{i=1}^{n} \ddot{l}_\theta(x_i)$$

$$\therefore \mathcal{E}(\hat{I}_n) = 0 - \mathcal{E}\left[\frac{1}{n} \sum_{i=1}^{n} \ddot{l}_\theta(x_i) \right]$$
$$= I_\theta$$

$\therefore \hat{I}_n$ is a consistent estimator of I_θ.

$\therefore$ by the continuous mapping theorem, $\hat{I}_n^{-1}$ is a consistent estimator of I_θ^{-1}

[Proved]

(b) $\sqrt{n}\, I_{\underset{\sim}{\theta}}^{1/2} (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}) \xrightarrow{d} N(0, I_{k\times k})$ [$\because$ all the regularity conditions are satisfied]

Now, $\hat{I}_n \xrightarrow{P} I_{\underset{\sim}{\theta}} \Rightarrow \dfrac{\hat{I}_n}{I_{\underset{\sim}{\theta}}} \xrightarrow{P} 1 \Rightarrow \dfrac{\hat{I}_n^{1/2}}{I_{\underset{\sim}{\theta}}^{1/2}} \xrightarrow{P} 1$ [by Cont. Mapping Thm]

by Slutsky's Theorem, $\sqrt{n}\, \hat{I}_n^{1/2} (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}) \xrightarrow{d} N(0, I_{k\times k})$

$\therefore\ n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})' \hat{I}_n (\underset{\sim}{\theta}_n - \underset{\sim}{\theta}) \xrightarrow{d} \chi^2_k$

So, the required $(1-\alpha)$ CI is given by:-

$\{\underset{\sim}{\theta} : n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})' \hat{I}_n (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}) < \chi^2_{\alpha,k}\}$ where $\chi^2_{\alpha,k}$ is the $(1-\alpha)^{th}$ percentile of χ^2_k distribution.

(c) $\sqrt{n}\, I_{\theta}^{1/2} (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}) \xrightarrow{d} N(0, I_{k\times k})$

$\Rightarrow \sqrt{n}\, I_{\xi_n}^{1/2} (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}) \xrightarrow{d} N(0, I_{k\times k})$ [if $\xi_n \to \theta$, by Slutsky's Thm]

$\Rightarrow n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})' I_{\xi_n} (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}) \xrightarrow{d} \chi^2_k \longrightarrow$ eq (i).

Now, $l_n(\underset{\sim}{\theta}) = \sum_{i=1}^{n} l_{\underset{\sim}{\theta}}(x_i)$

$= \sum_{i=1}^{n} l_{\hat{\theta}_n}(x_i) + \boxed{\cancel{\sum_{i=1}^{n} l_{\hat{\theta}_n}(x_i)}}\ (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})' \left(\sum_{i=1}^{n} \dot{l}_{\hat{\theta}_n}(x_i)\right)$

$+ \frac{1}{2}(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})' \left[\sum_{i=1}^{n} \ddot{l}_{\tilde{\underset{\sim}{\theta}}_n}(x_i)\right] (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})$, for some $\tilde{\theta}_n$

$\Rightarrow -2(l_n(\underset{\sim}{\theta}) - l_n(\hat{\underset{\sim}{\theta}}_n)) = (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})' \left(\sum_{i=1}^{n} \dot{l}_{\hat{\theta}_n}(x_i)\right)$

$+ n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})' I_{\tilde{\theta}_n} (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})$

$= 0 + n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})' I_{\tilde{\theta}_n} (\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})$

$\xrightarrow{d} \chi^2_k$ (by eq (i), as $\hat{\underset{\sim}{\theta}}_n \xrightarrow{a.s.} \underset{\sim}{\theta} \Rightarrow \tilde{\theta}_n \xrightarrow{a.s.} \theta$)

$\Rightarrow -2(l_n(\underset{\sim}{\theta}) - l_n(\hat{\underset{\sim}{\theta}}_n)) \xrightarrow{d} \chi^2_k$.

$\therefore$ the required $(1-\alpha)$ CI is given by:-

$\{\underset{\sim}{\theta} : -2(l_n(\underset{\sim}{\theta}) - l_n(\hat{\underset{\sim}{\theta}}_n)) < \chi^2_{\alpha,k}\}$

Question 2 (No 5 of Class Notes)

(a) We assume that the complete data is the genotype data which is a multinomial model with 6 possible outcomes (OO, AA, AO, BB, BO, AB) occurring with prob $(r^2, p^2, 2rp, q^2, 2rq, 2pq)$

Let us assume that the complete data has n_1 individuals with genotype AA & n_2 individuals with genotype BB

the complete log-likelihood is given by

$$l/n_1, n_2 = \text{constant} + N_O \log(r^2) + n_1 \log(p^2) + (N_A - n_1)\log(2rp)$$
$$+ n_2 \log(q^2) + (N_B - n_2)\log(2qr) + N_{AB} \log(2pq)$$
$$= k + (2N_O + N_A + N_B - n_1 - n_2)\log(r) + (n_1 + N_A + N_{AB})\log(p)$$
$$+ (n_2 + N_B + N_{AB})\log(q)$$

$$= K + a\log r + b\log p + c\log q \quad \text{[where } a, b \text{ \& } c \text{ depend on } n_1 \text{ \& } n_2\text{]}$$

$$[\ r = 1 - p - q\ ]$$

$$\frac{\partial l}{\partial p} = -\frac{a}{1-p-q} + \frac{b}{p}$$

$$\frac{\partial l}{\partial q} = -\frac{a}{1-p-q} + \frac{c}{q}$$

E step

For the E-step at the k-th step we must have

$$E\left[\frac{\partial l}{\partial p} \,\Big|\, p^{(k)}, q^{(k)}, r^{(k)}\right] = 0$$

$$\& \quad E\left[\frac{\partial l}{\partial q} \,\Big|\, p^{(k)}, q^{(k)}, r^{(k)}\right] = 0$$

Let $a_k = \mathcal{E}(a \mid p^{(k)}, q^{(k)}, r^{(k)})$, $b_k = \mathcal{E}(b \mid p^{(k)}, q^{(k)}, r^{(k)})$

& $c_k = \mathcal{E}(c \mid p^{(k)}, q^{(k)}, r^{(k)})$

$\therefore$ E-step yields :—

$$-\frac{a_k}{1-p-q} + \frac{b_k}{p} = 0 \longrightarrow eq\ (\text{i})$$

$$-\frac{a_k}{1-p-q} + \frac{c_k}{q} = 0 \longrightarrow eq\ (\text{ii})$$

$$a_k = \mathcal{E}\left[2N_O + N_A + N_B - n_1 - n_2 \mid p^{(k)}, q^{(k)}, r^{(k)} \right]$$

$$= 2N_O + N_A + N_B - \frac{N_A p^{(k)}}{p^{(k)} + 2r^{(k)}} - \frac{N_B q^{(k)}}{q^{(k)} + 2r^{(k)}}$$

$\left[\because n_1 / N_A, p^{(k)}, q^{(k)}, r^{(k)} \sim \text{Bin}\left(N_A, \frac{p^{(k)}}{p^{(k)} + 2r^{(k)}}\right) \right.$

$\left. \& \ n_2 / N_B, p^{(k)}, q^{(k)}, r^{(k)} \sim \text{Bin}\left(N_B, \frac{q^{(k)}}{q^{(k)} + 2r^{(k)}}\right) \right]$.

$$= 2\left[N_O + \frac{N_A r^{(k)}}{p^{(k)} + 2r^{(k)}} + \frac{N_B r^{(k)}}{q^{(k)} + 2r^{(k)}} \right.$$

Similarly, $b_k = \dfrac{N_A p^{(k)}}{p^{(k)} + 2r^{(k)}} + N_A + N_{AB}$

& $c_k = \dfrac{N_B q^{(k)}}{q^{(k)} + 2r^{(k)}} + N_B + N_{AB}$

<u>M-step</u>

Solving eq (i) & (ii) we get,

$$\hat{p} = \frac{b_k}{a_k + b_k + c_k} \qquad \hat{q} = \frac{c_k}{a_k + b_k + c_k}, \quad \hat{r} = \frac{a_k}{a_k + b_k + c_k}.$$

So, to apply the E-M algorithm, we start with an initial choice $p^{(0)}, q^{(0)}, r^{(0)}$ & compute p, q, r iteratively as follows :—

$$p^{(k+1)} = \frac{N_A\, p^{(k)} / \left(p^{(k)} + 2r^{(k)}\right) + N_A + N_{AB}}{2(N_O + N_A + N_B + N_{AB})}$$

$$q^{(k+1)} = \frac{N_B\, q^{(k)} / \left(q^{(k)} + 2r^{(k)}\right) + N_B + N_{AB}}{2(N_O + N_A + N_B + N_{AB})}$$

$$r^{(k+1)} = \frac{N_O + N_A\, r^{(k)} / \left\{p^{(k)} + 2r^{(k)}\right\} + N_B\, r^{(k)} / \left\{q^{(k)} + 2r^{(k)}\right\}}{N_O + N_A + N_B + N_{AB}}$$

Stopping condition

Fix some small $\varepsilon > 0$ & stop when $\max\{|p^{(k+1)} - p^{(k)}|, |q^{(k+1)} - q^{(k)}|, |r^{(k+1)} - r^{(k)}|\} < \varepsilon$ & conclude that $(p^{(k+1)}, q^{(k+1)}, r^{(k+1)})$ are the required EM-algorithm estimates of (p, q, r)

(b) We take $N_O = 176$, $N_A = 182$, $N_B = 60$ & $N_{AB} = 17$. We start with $p^{(0)} = q^{(0)} = r^{(0)} = 1/3$ & take $\varepsilon = 10^{-4}$. The iterations are shown below –

k	$p^{(k)}$	$q^{(k)}$	$r^{(k)}$	Max Diff
1	0.2985	0.1115	0.5900	0.2567
2	0.2710	0.0945	0.6346	0.0445
3	0.2655	0.0933	0.6412	0.0066
4	0.2646	0.0932	0.6422	0.0010
5	0.2645	0.0932	0.6424	0.0002
6	0.2644	0.0932	0.6424	$< 0.0001 = \varepsilon$.

the required estimates of (p, q, r) are $(0.2644, 0.0932, 0.6424)$

Question 3 (No 6 of Class Notes)

(a) X has density $f(x)$ & $Y/X=x \sim N(\beta x, \sigma^2)$. $(X_1, Y_1), \ldots (X_n, Y_n)$ are iid obs from (X,Y), but $X_{m+1} \ldots X_n$ are missing $1<m<n$ & the missingness satisfies MAR condition. The observed X_i's are distinct & we assume that X has point mass $P_i > 0$ at the observed data $X_i = x_i$, for $i = 1(1)m$, $\sum_{i=1}^{m} P_i = 1$

the likelihood function is given by

$$L = \prod_{i=1}^{m}\left[P_i \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(Y_i-\beta X_i)^2}{2\sigma^2}\right\}\right] \times \prod_{i=m+1}^{n}\left[\sum_{j=1}^{m} P_j \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(Y_i-\beta X_j)^2}{2\sigma^2}\right\}\right]$$

(b) $l_1 = \log(L_1) =$

$$\frac{\partial}{\partial\beta}(l_1) = 0 \Rightarrow \sum_{i=1}^{m} \frac{(Y_i-\beta X_i)X_i}{2\sigma^2} + \sum_{i=m+1}^{n} \frac{\sum_{j=1}^{m} P_j \exp\left\{-\frac{(Y_i-\beta X_j)^2}{2\sigma^2}\right\} X_j (Y_i - \beta X_j)}{\sum_{j=1}^{m} P_j \exp\left\{-\frac{(Y_i-\beta X_j)^2}{2\sigma^2}\right\}} = 0$$

$$\frac{\partial(l_1)}{\partial\sigma^2} = 0 \Rightarrow -\frac{n}{2\sigma^2} + \sum_{i=1}^{m} \frac{(Y_i-\beta X_i)^2}{2\sigma^4} + \sum_{i=m+1}^{n} \frac{\sum_{j=1}^{m} P_j \exp\left\{-\frac{(Y_i-\beta X_j)^2}{2\sigma^2}\right\} \frac{(X_i-\beta X_j)^2}{2\sigma^4}}{\sum_{j=1}^{m} P_j \exp\left\{-\frac{(Y_i-\beta X_j)^2}{2\sigma^2}\right\}} = 0$$

$$\frac{\partial(l_1)}{\partial P_a} = 0 \Rightarrow \frac{1}{P_a} - \frac{1}{1-\sum_{j=1}^{m-1} P_j} + \sum_{i=m+1}^{n} \frac{\left[\exp\left\{-\frac{(Y_i-\beta X_a)^2}{2\sigma^2}\right\} - \exp\left\{-\frac{(Y_i-\beta X_m)^2}{2\sigma^2}\right\}\right]}{\sum_{j=1}^{m} P_j \exp\left\{-\frac{(Y_i-\beta X_j)^2}{2\sigma^2}\right\}} = 0$$

$a = 1(1)m-1 \quad \left[\sum_{j=1}^{m} P_j = 1\right]$

(c) For the EM algorithm, we assume that the missing data $X_{m+1}, \ldots X_n$ is known. Then the complete likelihood is given by –

$$L_2 = \prod_{i=1}^{m}\left[P_i \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(Y_i-\beta X_i)^2}{2\sigma^2}\right\}\right] \times \prod_{i=m+1}^{n}\left[f_i \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(Y_i-\beta X_i)^2}{2\sigma^2}\right\}\right]$$

$$\therefore l_2 = \log(L_2) = \sum_{i=1}^{m} \log(P_i) - n\log(\sigma^2) - \sum_{i=1}^{m} \frac{(Y_i-\beta X_i)^2}{2\sigma^2} - \sum_{i=m+1}^{n} \frac{(Y_i-\beta X_i)^2}{2\sigma^2} + \sum_{i=m+1}^{n} \log(f_i) + \text{constant}$$

$$L_2 = \prod_{i=1}^{m} \left[\frac{P_i}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(y_i - \beta x_1)^2}{\sigma^2} \right\} \right] \prod_{i=m+1}^{n} \prod_{j=1}^{m} \left[p_j \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(y_i - \beta x_j)^2}{\sigma^2} \right\} \right]^{I_{ij}}$$

$I_{ij} = I\{x_i = x_j\}$ $\quad i = (m+1)(1)n, \; j = 1(1)m \quad \sum_{j=1}^{m} I_{ij} = 1.$

$$l_2 = \sum_{i=1}^{m} \log P_i - \frac{n}{2}\log(\sigma^2) - \sum_{i=1}^{m} \frac{(y_i - \beta x_1)^2}{\sigma^2} + \sum_{i=m+1}^{n} \sum_{j=1}^{m} I_{ij} \log(P_j)$$

$$- \sum_{i=m+1}^{n} \sum_{j=1}^{m} I_{ij} \frac{(y_i - \beta x_j)^2}{\sigma^2} + \text{const (indep of } \cancel{\text{other}} \text{ parameters)}$$

$$\frac{\partial l_2}{\partial p_j} = \frac{1}{P_j} + \frac{1}{P_j} \sum_{i=m+1}^{n} I_{ij} - \lambda \quad \left[\text{Using Lagrange multipliers since } \sum_{i=1}^{m} P_i = 1\right]$$

$$\frac{\partial l_2}{\partial \beta} = \frac{\sum_{i=1}^{m} (y_i - \beta x_1) x_i + \sum_{i=m+1}^{n} \sum_{j=1}^{m} (y_i - \beta x_j) x_j}{2\sigma^2}$$

$$\frac{\partial l_2}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \left[\sum_{i=1}^{m} (y_i - \beta x_i)^2 + \sum_{i=m+1}^{n} \sum_{j=1}^{m} I_{ij} (y_i - \beta x_j)^2 \right]$$

<u>E-step</u>

$$I_{ij}^{(k)} = E\left[I_{ij} / p_1^{(k)}, \ldots, p_m^{(k)}, \underset{\sim}{y}, \beta^{(k)}, \sigma^{2(k)} \right]$$

$$= P\left(x_i = x_j / P_1^{(k)}, \ldots, P_m^{(k)}, \underset{\sim}{y}, \beta^{(k)}, \sigma^{2(k)} \right)$$

$$= \frac{P_j^{(k)} \frac{1}{\sqrt{2\pi\sigma^{2(k)}}} \exp\left\{ -\frac{(y_i - \beta^{(k)} x_j)^2}{\sigma^{2(k)}} \right\}}{\sum_{l=1}^{m} P_l^{(k)} \frac{1}{\sqrt{2\pi\sigma^{2(k)}}} \exp\left\{ -\frac{(y_i - \beta^{(k)} x_l)^2}{\sigma^{2(k)}} \right\}}$$

$$= \frac{P_j^{(k)} \exp\left\{ -\frac{(y_i - \beta^{(k)} x_j)^2}{\sigma^{2(k)}} \right\}}{\sum_{l=1}^{m} P_l^{(k)} \exp\left\{ -\frac{(y_i - \beta^{(k)} x_l)^2}{\sigma^{2(k)}} \right\}}$$

So, the score equations at the $(k+1)$-th step are: $\rightarrow$

$$\cancel{\mathcal{E}\left[\frac{\partial l_2}{\partial p_i}\right] = 0}$$

$$\Rightarrow \mathcal{E}\left[\frac{\partial l_2}{\partial p_j} \,\Big|\, p_1^{(k)}, \ldots, p_m^{(k)}, \underset{\sim}{y}, \beta^{(k)}, \sigma^{2(k)}\right] = 0$$

$$\Rightarrow \frac{1}{p_j} + \frac{1}{p_j} \sum_{i=m+1}^{n} I_{ij}^{(k)} - \lambda = 0 \longrightarrow (i)$$

$$\mathcal{E}\left[\frac{\partial l_2}{\partial \beta} \,\Big|\, p_1^{(k)}, \ldots, p_m^{(k)}, \underset{\sim}{y}, \beta^{(k)}, \sigma^{2(k)}\right] = 0$$

$$\Rightarrow \sum_{i=1}^{m} (y_i - \beta x_i) x_i + \sum_{i=m+1}^{n} \sum_{j=1}^{m} I_{ij}^{(k)} (y_i - \beta x_j) x_j = 0 \longrightarrow (ii)$$

$$\mathcal{E}\left[\frac{\partial l_2}{\partial \sigma^2} \,\Big|\, p_1^{(k)}, \ldots, p_m^{(k)}, \underset{\sim}{y}, \underset{\sim}{\beta}^{(k)}, \sigma^{2(k)}\right] = 0$$

$$\Rightarrow -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\left[\sum_{i=1}^{m} (y_i - \beta x_i)^2 + \sum_{i=m+1}^{n} \sum_{j=1}^{m} I_{ij}^{(k)} (y_i - \beta x_j)^2\right] = 0 \longrightarrow (iii)$$

M - step

Solving eq (i), (ii) & (iii) we get,

$$p_j^{(k+1)} = \frac{1 + \sum_{i=m+1}^{n} I_{ij}^{(k)}}{n}$$

$$\hat{\beta}^{(k+1)} = \frac{\sum_{i=1}^{m} x_i y_i + \sum_{i=m+1}^{n} \sum_{j=1}^{n} I_{ij}^{(k)} y_i x_j}{\sum_{i=1}^{m} x_i^2 \left(1 + \sum_{j=m+1}^{n} I_{ji}^{(k)}\right)}$$

$$\hat{\sigma}^{2(k+1)} = \frac{1}{n}\left[\sum_{i=1}^{m} (y_i - \hat{\beta}^{(k+1)} x_i)^2 + \sum_{i=m+1}^{n} \sum_{j=1}^{m} I_{ij}^{(k)} (y_i - \hat{\beta}^{(k+1)} x_j)^2\right]$$

Stopping Condition

We fix some small $\varepsilon > 0$ & stop when $\max\left\{ |\hat{\beta}^{(k+1)} - \hat{\beta}^{(k)}|, |\hat{\sigma}^{2(k+1)} - \hat{\sigma}^{2(k)}|, |\hat{p}_1^{(k+1)} - \hat{p}_1^{(k)}|, \ldots, |\hat{p}_m^{(k+1)} - \hat{p}_m^{(k)}| \right\} < \varepsilon$ & conclude that $(\hat{\beta}^{(k+1)}, \hat{\sigma}^{2(k+1)}, \hat{p}_1^{(k+1)}, \ldots, \hat{p}_m^{(k+1)})$ are the required estimates using the EM algorithm.