BIOS 760 HOMEWORK I SOLUTION

1. (a) $Y_1, \ldots, Y_{n+1}$ are independent with $\exp(\theta)$. Let $U = Y_1 + \ldots + Y_i, V = Y_{i+1} + \ldots + Y_{n+1}$. Then $U \sim \text{Gamma}(i, \theta), V \sim \text{Gamma}(n + 1 - i, \theta)$. Let

$$Z_i = U/(U + V), W = U + V.$$ 

Consider the transformation $(U, V)' \mapsto (Z_i, W)'$. Note that the transformation is one-to-one with the Jacobian

$$|\det \left( \frac{\partial(U, V)}{\partial(Z_i, W)} \right) | = |\det \left( \begin{pmatrix} W & Z_i \\ -W & 1 - Z_i \end{pmatrix} \right) | = |W|.$$

From the joint density of $(U, V)'$,

$$\frac{1}{\Gamma(i)} \theta \exp\{-\theta u\}(\theta u)^{i-1} I(u > 0) \times \frac{1}{\Gamma(n + 1 - i)} \theta \exp\{-\theta v\}(\theta v)^{n-i} I(v > 0),$$

we obtain the joint density of $(Z_i, W)$ as

$$\frac{1}{\Gamma(i)} \theta \exp\{-\theta z_i w\}(\theta z_i w)^{i-1} \times \frac{1}{\Gamma(n + 1 - i)} \theta \exp\{-\theta(1 - z_i) w\}(\theta(1 - z_i) w)^{n-i} w$$

$$\times I(0 < z_i < 1) I(w > 0),$$

Thus, the marginal density of $Z_i = X/(X + Y)$ is equal to

$$(1 - z_i)^{n-i} z_i^{i-1} I(0 < z_i < 1) \frac{\Gamma(n + 1)}{\Gamma(i) \Gamma(n + 1 - i)} \int_w \theta \exp\{-\theta w\}(\theta w)^{n} I(w > 0) dw$$

$$= \frac{\Gamma(n + 1)}{\Gamma(i) \Gamma(n + 1 - i)} z_i^{i-1}(1 - z_i)^{n-i} I(0 < z_i < 1).$$

That is, $Z_i \sim \text{Beta}(i, n + 1 - i)$.

(b) Let

$$W_1 = Y_1, W_2 = Y_1 + Y_2, \ldots, W_n = Y_1 + \ldots + Y_n, S = Y_1 + \ldots + Y_{n+1}.$$ 

Consider the transformation $(Y_1, \ldots, Y_{n+1})' \mapsto (W_1, \ldots, W_n, S)'$. Note that the transformation is one-to-one with the Jacobian

$$|\det \left( \frac{\partial(Y_1, \ldots, Y_{n+1})}{\partial(W_1, \ldots, W_n, S)} \right) | = |\det \left( \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -1 & 1 \end{pmatrix} \right) | = 1$$

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From the joint density of \((Y_1, ..., Y_{n+1})'\),
\[
\prod_{i=1}^{n+1} \frac{1}{\theta} \exp\{-\frac{1}{\theta}y_i\}I(0 < y_i < \infty) = \left(\frac{1}{\theta}\right)^{n+1} \exp\{-\frac{\sum_{i=1}^{n+1} y_i}{\theta}\}I(0 < y_i < \infty),
\]
we obtain the joint density of \((W_1, ..., W_n, S)\) as
\[
\left(\frac{1}{\theta}\right)^{n+1} \exp\{-\frac{s}{\theta}\}I(0 < w_1 < w_2 < ... < w_n < s < \infty),
\]
Let
\[
Z_1 = W_1/S, Z_2 = W_2/S, ..., Z_n = W_n/S, S.
\]
Consider the transformation \((W_1, ..., W_n, S)' \mapsto (Z_1, ..., Z_n, S)'\). Note that the transformation is one-to-one with the Jacobian
\[
|\det \left( \frac{\partial(W_1, ..., W_n, S)}{\partial(Z_1, ..., Z_n, S)} \right) | = |\det \left( \begin{array}{cccc}
 s & 0 & \ldots & 0 \\
 0 & s & \ldots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \ldots & s \\
 0 & 0 & \ldots & 1 \\
\end{array} \right) | = s^n
\]
From the joint density of \((Z_1, ..., Z_n, S)'\),
\[
= \left(\frac{1}{\theta}\right)^{n+1} \exp\{-\frac{s}{\theta}\} s^nI(0 < z_1 < z_2 < ... < z_n < 1)I(s > 0)
\]
We obtain the joint density of \((Z_1, ..., Z_n)'\),
\[
f(z_1, ..., z_n) = \int_s f(z_1, ..., z_n, S)ds = n!I(0 < z_1 < z_2 < ... < z_n < 1)
\]
which is the joint density of order statistics of \(n\) uniform(0,1) random variables.

2. \(\text{Cov}(X,Y|Z) = E(XY|Z) - E(X|Z)E(Y|Z)\), thus
\[
E[\text{Cov}(X,Y|Z)] = E(XY) - E[E(X|Z)E(Y|Z)].
\]
Since
\[
\text{Cov}(E(X|Z), E(Y|Z)) = E[E(X|Z)E(Y|Z)] - E[E(X|Z)]E[E(Y|Z)]
\]
\[
= E[E(X|Z)E(Y|Z)] - E(X)E(Y),
\]
we have
\[
E[\text{Cov}(X,Y|Z)] + \text{Cov}(E(X|Z), E(Y|Z)) = E(XY) - E(X)E(Y) = \text{Cov}(X,Y)
\]
3. (a) For \( z < 0 \), \( P(Z \leq z, \Delta = 1) = P(Z \leq z, \Delta = 0) = 0 \). For \( z \geq 0 \),
\[
P(Z \leq z, \Delta = 1) = P(X \leq Y, X \leq z) = E[I(X \leq Y, X \leq z)] = E[E[I(X \leq Y, X \leq z)|X]] = E[I(X \leq z)(1 - G(X-))] = \int_0^z (1 - G(x-))dF(x).
\]
By the symmetry, \( P(Z \leq z, \Delta = 0) = \int_0^z (1 - F(x-))dG(x) \).

(b) Note
\[
\int_0^z (1 - G(x-))dF(x) = \int_0^z \lambda \exp{-\mu x} \exp{-\lambda x}dx = \frac{\lambda}{\lambda + \mu} (1 - \exp{-(\lambda + \mu)z})
\]
and
\[
\int_0^z (1 - F(x-))dG(x) = \int_0^z \mu \exp{-\mu x} \exp{-\lambda x}dx = \frac{\mu}{\lambda + \mu} (1 - \exp{-(\lambda + \mu)z}).
\]
In other words,
\[
P(Z \leq z, \Delta = \delta) = (1 - \exp{-(\lambda + \mu)z}) \left\{ \frac{\lambda}{\lambda + \mu} \right\}^\delta \left\{ \frac{\mu}{\lambda + \mu} \right\}^{1-\delta}.
\]
Thus \( Z \) and \( \Delta \) are independent.

4. (a) Let \( \Sigma \) be an orthogonal matrix with the first row being \((\sqrt{\omega_1}, ..., \sqrt{\omega_n})\). Define \((Z_1, ..., Z_n)^T = \Sigma(X_1, ..., X_n)^T\). Then \((Z_1, ..., Z_n)^T\) follows a multivariate normal distribution with mean zeros and covariance \( \sigma^2 I_{n \times n} \). Note \( Y_n = Z_1/\sigma \). Thus, \( Y_n \sim N(0,1) \).

(b) Note \( Z_1^2 + ... + Z_n^2 = X_1^2 + ... + X_n^2 \). Thus,
\[
(n - 1)S_n^2/\sigma^2 = \left( \sum_{i=1}^n Z_i^2 - Z_1^2 \right)/\sigma^2 = \sum_{i=2}^n Z_i^2/\sigma^2 \sim \chi_n^2.
\]

(c) Since \( Z_1 \) is independent of \( Z_2, ..., Z_n \), so are \( \bar{X}_{nw} \) and \( S_n^2 \). \( T_n \sim N(0,1)/\sqrt{S_n^2} = t_{n-1}/\sigma \).

(d) When \( \omega_1 = ... = \omega_n = 1/n \), \( Y_n = \sqrt{n}\bar{X}_{n}/\sigma \) and \( S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2 \). We obtain the result.
5. (a) \( P(W_m = k) = \binom{k-1}{m-1} p^m (1 - p)^{k-m} = \binom{k-1}{m-1} \exp\{k \log(1 - p) + m \log(p/(1 - p))\} \).

This is a 1-parameter exponential family with \( T(x) = x, \eta = \log(1 - p) \) and \( A(\eta) = -m \log(p/(1 - p)) = m \log\{e^\eta/(1 - e^\eta)\} \).

(b) The moment generating function for \( W_m \), equivalently \( T \), is
\[
M_T(t) = \exp\{A(\eta + t) - A(\eta)\} = e^{mt} \frac{(1 - e^\eta)^m}{(1 - e^{\eta+t})^m} = e^{mt} \frac{p^m}{(1 - (1 - p)e^t)^m}.
\]

(c) The cumulant generating function for \( W_m \) is equal to
\[
mt + m \log p - m \log(1 - (1 - p)e^t).
\]

After differentiation, we obtain the first two cumulants as
\[
\mu_1 = p, \quad \mu_2 = \frac{m}{p^2} - \frac{m}{p}.
\]