Solution to Midterm Exam II

1. By SLLN,

$$n^{-1}\sum_{i=1}^{n} X_iY_i \to_{a.s} E[XY] = \beta E[X^2], \quad n^{-1}\sum_{i=1}^{n} X_i^2 \to_{a.s} E[X^2].$$

Thus, $\hat{\beta} \to_{a.s.} \beta$ from the continuous mapping theorem.

2. By the CLT,

$$n^{-1/2}\sum_{i=1}^{n} X_i(Y_i - \beta X_i) \to_d N(0, E[X^2]Var(\epsilon)).$$

From the Slutsky lemma,

$$\sqrt{n}(\hat{\beta} - \beta) = \frac{n^{-1/2}\sum_{i=1}^{n} X_i\epsilon_i}{n^{-1}\sum_{i=1}^{n} X_i^2} \to_d N\left(0, Var(\epsilon)/E[X^2]\right).$$

3. Note

$$T_n = (\hat{\beta} - \beta)\left\{-n^{-1}\sum_{i=1}^{n} Y_iX_i + (\hat{\beta} + \beta)n^{-1}\sum_{i=1}^{n} X_i^2\right\}.$$

By the Slutsky lemma,

$$\sqrt{n}T_n \to \beta E[X^2]N(0, Var(\epsilon)/E[X^2]).$$

4. (a) It follows from the continuous mapping theorem with $g(x) = \max(x, 0)$.

   (b) Note

$$P\left(\sqrt{n}(\tilde{\beta} - \beta) \le x\right) = P(\sqrt{n}(-\beta) \le x, \hat{\beta} < 0) + P(\sqrt{n}(\hat{\beta} - \beta) \le x, \hat{\beta} \ge 0).$$

Since $P(\sqrt{n}(-\beta) \le x, \hat{\beta} < 0) = 0$ for large n,

$$P\left(\sqrt{n}(\tilde{\beta} - \beta) \le x\right) = P(\sqrt{n}(\hat{\beta} - \beta) \le x, \hat{\beta} \ge 0)$$

$$= P(\sqrt{n}(\hat{\beta} - \beta) \le x) - P(\sqrt{n}(\hat{\beta} - \beta) \le x, \hat{\beta} < 0).$$

Note

$$P(\sqrt{n}(\hat{\beta} - \beta) \le x, \hat{\beta} < 0) \le P(\hat{\beta} < 0) \le P(|\hat{\beta} - \beta| > \beta) \to 0.$$

Thus,

$$|P\left(\sqrt{n}(\tilde{\beta}-\beta)\le x\right)-P(\sqrt{n}(\hat{\beta}-\beta)\le x)|\to 0.$$

That is, the limiting distribution is the same as $\sqrt{n}(\hat{\beta}-\beta)$.

(c) When $\beta=0$,

$$\sqrt{n}\tilde{\beta}=\max(\sqrt{n}\hat{\beta},0)\to_d \max\left\{N(0,\left(0,Var(\epsilon)/E[X^2]\right)),0\right\}$$

by the continuous mapping theorem.

5. (a) $1-F(x)=P(|X_1|>x)\le m_6/x^6$.

(b) $P(\max_i|X_i|>\sqrt{n}\delta)=1-\prod_i P(|X_i|\le\sqrt{n}\delta)=1-F(\sqrt{n}\delta)^n\le 1-(1-m_6/n^3\delta^6))^n$. Since $(1-x)^n>e^{-nx}$,

$$\sum_n P(\max_i|X_i|>\sqrt{n}\delta)\le\sum_n(1-e^{-m_6/n^2\delta^6})\le\sum_n m_6/n^2\delta^6.$$

The result holds.

(c) It follows from the first Borel-Cantelli lemma that $\max_i|X_i|/\sqrt{n}<\delta$ with probability one for large n and for any $\delta>0$. That is, $\max_i|X_i|/\sqrt{n}\to_{a.s.}0$.

(d) Note

$$\sqrt{n}(\hat{\beta}-\beta)=\sum_{i=1}^n\omega_{ni}\epsilon_i,$$

where $\omega_{ni}=n^{-1/2}X_i/(\sum_{i=1}^n X_i^2/n)$. Clearly, $\max_i|\omega_{ni}|\equiv\max_i\sqrt{n}|X_i|/(\sum_{i=1}^n X_i^2/n)\to_{a.s.}0$. The Lindeberg-Feller condition can be verified using the same arguments as proving the weighted CLT, except that the arguments are condition on $X_1,X_2,....$ Particularly, $\sigma_n^2=n\sigma_y^2/\sum_i X_i^2$ with $\sigma_y^2=Var(\epsilon)$ and for any $\eta>0$,

$$\frac{1}{\sigma_n^2}\sum_{i=1}^n E[\omega_{ni}^2\epsilon_i^2 I(\omega_{ni}|\epsilon_i|>\eta\sigma_n)|X_1,...,X_n]$$

$$\le\frac{1}{\sigma_n^2}\sum_{i=1}^n E[\omega_{ni}^2\epsilon_1^2 I(|\epsilon_1|>\eta\sigma_n/\max_i|\omega_{ni}|)|X_1,...,X_n]$$

$$=\sigma_y^{-2}E[\epsilon_1^2 I(|\epsilon_1|>\eta\sigma_n/\max_i|\omega_{ni}|)]\to_{a.s}0.$$

From the Lindeberg-Feller CLT,

$$\sigma_n^{-1} \sum_{i=1}^{n} \omega_{ni} \epsilon_i \to_d N(0,1).$$

Since $\sigma_n^2 \to_{a.s.} \sigma_y^2/E[X^2]$, we conclude that condition on $X_1, ..., X_n$,

$$\sqrt{n}(\hat{\beta} - \beta) \to_d N(0, \sigma_y^2/E[X^2]).$$