Solution to Midterm Exam I

1. (a) Since

$$Var(Y - \gamma^T X) = \Sigma_{11} + \gamma^T \Sigma_{22}\gamma - 2\Sigma_{12}\gamma,$$

the γ minimizing this expression is

$$\gamma_m = \Sigma_{22}^{-1}\Sigma_{12}$$

and the minimal variance $\sigma_m^2 = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$.

(b) Note $Cov(Y - \gamma_m^T X, X) = 0$. Moreover, $(Y - \gamma_m^T X)^2/\sigma^2 \sim \chi_2(1)$ and $X^T \Sigma_{22}^{-1} X^T \sim \chi_2(k)$. Thus, the distribution of the ratio is $F_{1,k}/k$.

2. (a) Since for any Borel set $B, B_1, B_2, ...$, $Y^{-1}(B)^c = Y^{-1}(B^c) \in \mathcal{C}$ and $\cup_n Y^{-1}(B_n) = Y^{-1}(\cup_n B_n) \in \mathcal{C}$, $\mathcal{C}$ is a σ-field.

(b) Clearly, $\mu(Y^{-1}(B)) \geq 0$. Since $Y^{-1}(R) = \Omega$, $\mu(\phi) = \mu(Y^{-1}(\phi)) = \lambda(\phi) = 0$. Moreover, we note that if $Y^{-1}(B_1) \cap Y^{-1}(B_2) = \phi$, then $Y^{-1}(B_1 \cap B_2) = \phi$ so $B_1 \cap B_2 = \phi$. Thus, for any countable and disjoint sets in $\mathcal{C}$, say, $Y^{-1}(B_1), Y^{-1}(B_2), ...$, we have $B_1, B_2, ...$ are disjoint sets in R. As the result,

$$\mu(\cup_n Y^{-1}(B_n)) = \mu(Y^{-1}(\cup_n B_n)) = \lambda(\cup_n B_n) = \sum_n \lambda(B_n) = \sum_n \mu(Y^{-1}(B_n)).$$

Thus, μ is a measure in $\mathcal{C}$.

(c) Note $P(Y^{-1}(B)) = P(Y \in B) = \int_B d\mu_Y(y) = \int_B f(y) d\lambda(y)$. Thus, if $\mu(Y^{-1}(B)) = 0$, i.e., $\lambda(B) = 0$, then $P(Y^{-1}(B)) = 0$. By definition, P is denominated by μ.

(d) To find the derivative, we need determine a measurable function $g(\omega)$ in $(\omega, \mathcal{C})$ such that

$$P(Y^{-1}(B)) = \int_{Y^{-1}(B)} g(\omega) d\mu(\omega). \quad (1)$$

Note

$$P(Y^{-1}(B)) = \int_B f(y) d\lambda(y).$$

For any simple function $g(\omega) = \sum x_i I_{Y^{-1}(A_i)}(\omega)$,

$$\int_{Y^{-1}(B)} g(\omega) d\mu(\omega) = \sum_i x_i \lambda(A_i \cap B) = \int_B \sum_i x_i I_{A_i}(y) d\lambda(y) = \int_B g(Y^{-1}(y)) d\lambda(y).$$

Using the simple function to approximate any positive measurable function, we obtain that for any positive measurable function g,

$$\int_{Y^{-1}(B)} g(\omega) d\mu(\omega) = \int_B g(Y^{-1}(y)) d\lambda(y).$$

From equation (1), we conclude

$$\frac{dP(\omega)}{d\mu} = f(Y(\omega)).$$