Solution to Practice Problems for Midterm 2007

1. (a) Only need to show $\{\omega : X(\omega) \le x\} \in \mathcal{B}$ for any x and $\mu_F(\{\omega : X(\omega) \le x\}) = F(x)$. They are easy to verify.

   (b) Verify the same conditions. Use the property of the quantile function.

2. Clearly, $(A_n, B_n) \to_p (0, 0)$. From the Slutsky's theorem,

$$(X_n, Y_n) + (A_n, B_n) \to_d (X, Y).$$

I.e., $(X_n + A_n, Y_n + B_n) \to_d (X, Y)$. Apply the continuous mapping theorem to $g(x, y) = x/y$. Since $Y > 0$, we obtain the result. The counter example is as follows: let $X_n = X$ and $Y_n = X$ where X has a chi-square distribution. Let $\tilde{X}$ be another random variable indepdent of X but with the same distribution. Set $A_n = B_n = 0$. Then $X_n \to_d X$ and $Y_n \to_d \tilde{X}$ but

$$\frac{X_n}{Y_n} = 1 \text{ does not converge in distribution to } \frac{X}{\tilde{X}}.$$

3. (a) The proof that $\mathcal{B} \cap \Omega$ is closed under complement and countable union is as follows: if $B \cap \Omega$ is in the class, then

$$(B \cap \Omega)^c[\text{note this is the complement in } \Omega] = B^c \cap \Omega$$

   is also in since $B^c \in \mathcal{B}$; if $B_1 \cap \Omega, B_2 \cap \Omega, \ldots$ are in the class, then

$$\cup_n \{B_n \cap \Omega\} = \{\cup_n B_n\} \cap \Omega$$

   is in since $\cup_n B_n \in \mathcal{B}$. Obviously, $\lambda \times \lambda(\Omega) = 1$.

   (b) For any $z \in R$,

$$\{(x, y) : Z(x, y) \le z\} = \{(x, y) : y/x \le z\} \cap \Omega.$$

   Since $\{(x, y) : y/x \le z\}$ is in $\mathcal{B}$, $\{(x, y) : Z(x, y) \le z\}$ is in $\mathcal{B} \cap \Omega$. Thus, Z is measurable.

   (c) μ_Z is the Lebesgue-Stieljes measure generated by the distribution function of Z, $F_Z(z)$. Note that for $z \le 0$, $F_Z(z) = 0$; for $z > 0$,

$$F_Z(z) = \lambda \times \lambda \{(x, y) : y \le xz, (x, y) \in \Omega\} = \int_0^1 \int_0^1 I(y \le xz) dy dx = \int_0^1 \min(1, xz) dx.$$

   The latter is equal to $z/2$ if $z \le 1$ and is equal to $1 - 1/(2z)$ if $z > 1$.

   (d) Since F_Z has no discontinuous point, the dominating measure is the Lebesgue measure (this requires verifying the condition of absolute continuity). The density is

$$f_Z(z) = \frac{1}{2} I(0 < z \le 1) + \frac{1}{2z^2} I(z > 1).$$

   (e) From the density, we obtain

$$E[Z] = \int_0^1 \frac{z}{2} dz + \int_1^\infty \frac{1}{2z} dz = \infty.$$

   (f) Define a new random variable Y as $Y(x, y) = y$. One can easily check that Y and W are independent and both have uniform distribution in $[0, 1]$. Clearly, $Z = Y/W$. Thus,

$$E[Z|W] = E[Y/W|W] = E[Y|W]/W = E[Y]/W = 1/(2W).$$

   An alternative way is to find the joint density of (W, Z) then compute the conditional expectation.

4. (a) Since $Cov(X, Y - cX) = Cov(X, Y) - cCov(X, X) = \rho - c$, we conclude that if $c = \rho$, X and $Y - \rho X$ are independent.

   (b) Let $Z = Y - \rho X$.The calculation is the following:

$$E[X^2Y^2] = E[X^2(\rho X + Z)^2] = \rho^2 E[X^4] + 2\rho E[X^3 Z] + E[X^2 Z^2].$$

   Since X and Z are independent and with normal distributions with mean zeros, moreover, Z's variance is equal to $(1 - \rho^2)$, we obtain

$$E[X^2Y^2] = 3\rho^2 + (1 - \rho^2) = 1 + 2\rho^2.$$

   (c) The MGF of (X, Y) is given by

$$m(t, s) = \exp\left\{(t, s)\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}(t, s)'/2\right\} = \exp\left\{t^2/2 + s^2/2 + \rho ts\right\}.$$

   By differentiation, the coefficient of $t^2s^2/(2!)^2$ in the Taylor expansion of $m(t, s)$ is given by $(1 + 2\rho^2)$.

5. (a) The joint distribution is as follows: for $x, y \in (0, 1)$ and $x \le y$,

$$P(X_{(1)} > x, X_{(n)} \le y) = P(x < X_1 \le y, ..., x < X_n \le y) = (y - x)^n.$$

   Thus the joint density is equal to $n(n-1)(y-x)^{n-2}I(0 < x < 1, x \le y < 1)$.

   (b) $E[X_{(1)}|X_{(n)} = y] = \int_x x(y-x)^{n-2}I(0 < x < 1, x \le y < 1)dx / \int_x (y-x)^{n-2}I(0 < x < 1, x \le y < 1)dx = y/n$.

   (c) Using the transformation from $(X_{(1)}, X_{(n)})$ to $(X_{(1)} = x, X_{(n)} - X_{(1)} = z)$, we obtain that the latter has a joint density

$$n(n-1)z^{n-2}I(0 < x < 1, 0 \le z < 1 - x).$$

   After integrating out x, the density of $(X_{(n)} - X_{(1)})$ is equal to $n(n-1)z^{n-2}(1-z)I(0 \le z < 1)$, i.e., Beta-distribution $Beta(n-1, 2)$.

6. (a) Since $V = \{|U| + X + Y\}/2$, we obtain

$$X = V + (-|U| + U)/2, \quad Y = V + (-|U| - U)/2.$$

   (b) The support for (X, Y) is $\{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$. Thus, the support of (U, V) is

$$0 \le v + (-|u| + u)/2 \le 1, \quad 0 \le v + (-|u| - u)/2 \le 1.$$

   Clearly, $|u| \le 1$. Then if $u < 0$, the above inequalities give $-u \le v \le 1$; if $u \ge 0$, we obtain $u \le v \le 1$. Hence, the support of (U, V) should be $\{(u, v) : |u| \le v \le 1\}$.

   (c) Note $V = \{|X - Y| + (X + Y)\}/2$. Thus, the Jacobian of (U, V) with respect to (X, Y) is given by

$$\left|\begin{pmatrix} 1 & -1 \\ (sgn(x-y)+1)/2 & (-sgn(x-y)+1)/2 \end{pmatrix}\right| = 1.$$

   Then the Jacobian $\partial(X, Y)/\partial(U, V)$ is equal to 1. From the previous support calculation, we conclude that the joint density of (U, V) is equal to

$$f(u, v) = I\{|u| \le v \le 1\}.$$

(d) Obviously, U and V are not independent but since $E[U|V] = 0$,

$$Cov(U, V) = E[UV] - E[U]E[V] = 0.$$

(e) The marginal density of V is equal to $2vI(0 \leq v \leq 1)$. Thus,

$$E[U^2|V = 0.5] = \frac{\int_u u^2 I\{|u| \leq 1, |u| \leq v \leq 1\}}{2vI(0 \leq v \leq 1)} du\Big|_{v=0.5} = \frac{1}{12}.$$

7. (a) From $Cov(\tilde{\epsilon}_i, X_i) = \rho\sigma - c$, we conclude that if $c = \rho\sigma$, $\tilde{\epsilon}_i$ is independent of X_i.

(b) The expectation $\hat{\beta}_n$ is $\beta_0 + \rho\sigma$. This is from the following calculation

$$E[\hat{\beta}_n] = E\left[\frac{\sum_{i=1}^n X_i((\beta_0 + \rho\sigma)X_i + \tilde{\epsilon}_i)}{\sum_{i=1}^n X_i^2}\right] = E\left[\beta_0 + \rho\sigma + \frac{\sum_{i=1}^n X_i\tilde{\epsilon}_i}{\sum_{i=1}^n X_i^2}\right]$$

$$= \beta_0 + \rho\sigma + E\left[E\left[\frac{\sum_{i=1}^n X_i\tilde{\epsilon}_i}{\sum_{i=1}^n X_i^2}\Big|X_1, ..., X_n\right]\right] = \beta_0 + \rho\sigma.$$

Clearly, $\hat{\beta}_n$ is an unbiased estimate of β_0 if and only if $\rho\sigma = 0$, i.e., X_i and ϵ_i are independent.

(c) From the previous part, we know

$$\hat{\beta}_n = \beta_0 + \rho\sigma + \frac{\sum_{i=1}^n X_i\tilde{\epsilon}_i}{\sum_{i=1}^n X_i^2} \equiv \beta_0 + \rho\sigma + \frac{Q_n}{\sqrt{P_n}},$$

where

$$Q_n = \frac{\sum_{i=1}^n X_i\tilde{\epsilon}_i}{\sqrt{\sum_{i=1}^n X_i^2}}, \quad P_n = \sum_{i=1}^n X_i^2.$$

Note that conditional on $X_1, ..., X_n$,

$$Q_n = \frac{\sum_{i=1}^n X_i\tilde{\epsilon}_i}{\sqrt{\sum_{i=1}^n X_i^2}} \sim N(0, (1 - \rho^2)\sigma^2).$$

Thus, Q_n is independent of $(X_1, ..., X_n)$ so is independent of P_n. Moreover, P_n has a Chi-square distribution with n degrees of freedom. Then

$$\frac{Q_n}{P_n} \sim \sqrt{\frac{(1 - \rho^2)\sigma^2}{n}} t(n)$$

where $t(n)$ denotes the t-distribution with n degrees of freedom. Hence,

$$\hat{\beta}_n \sim \beta_0 + \rho\sigma + \sqrt{\frac{(1 - \rho^2)\sigma^2}{n}} t(n),$$

which is a shifted and scaled t-distribution with n degrees of freedom.

8. (a) The (X, Y)-induced measure, $\mu_{(X,Y)}$, is the Lebesgue-Stieltjes measure generated by the joint distribution of (X, Y). The joint distribution function of (X, Y) is the given by $\Phi(x)\Phi(y)$, where Φ is the cumulative normal distribution function. Specifically, for any Borel set B in R^2,

$$\mu_{(X,Y)}(B) = \int_{(x,y)\in B} \phi(x)\phi(y)d\lambda(x)d\lambda(y).$$

(b) Since $\{Z \le z\} = \{X \le z\} \cap \{Y \le z\}$, the conclusion is clear.

(c) Denote $E[X|Z] = g(Z)$. From the equation

$$E[I(Z \le z)g(Z)] = E[I(Z \le z)X] = E[XI(X \le z)I(Y \le z)],$$

we obtain

$$\int_{-\infty}^{z} g(z) f_Z(z) dz = \int_{-\infty}^{z} x\phi(x)dx\Phi(z),$$

where $f_Z(z)$ is the density of Z given by $d/dz(\Phi(z)^2)$. Differentiating both sides with respect to z, we have

$$\begin{aligned} g(z) &= \frac{z\phi(z)\Phi(z) + \int_{-\infty}^{z} x\phi(x)dx\phi(z)}{2\phi(z)\Phi(z)} \\ &= \frac{z}{2} + \frac{1}{2}\frac{\int_{-\infty}^{z} x\phi(x)dx}{\Phi(z)}. \end{aligned}$$

9. It suffices to show that for any subsequence of X_n, there exists a further subsequence such that $E[|X_n - X|] \to 0$. First, since $X_n \to_p X$, there exists a further subsequence such that $X_n \to_{a.s.} X$, still denoted as X_n. Apply the Fatou'e lemma to $|X_n| + |X| - |X_n - X|$ then we obtain

$$E\left[\liminf_n \{|X_n| + |X| - |X_n - X|\}\right] \le \liminf_n \{E[|X_n|] + E[|X|] - E[|X_n - X|]\}.$$

The left-hand side is equal $2E[|X|]$ while the right-hand side is equal to $2E[|X|] - \limsup_n E[|X_n - X|]$. Thus,

$$\limsup_n E[|X_n - X|] \le 0.$$

The conclusion holds.