1. (a) Only need to show \( \{ \omega : X(\omega) \leq x \} \in \mathcal{B} \) for any \( x \) and \( \mu_F (\{ \omega : X(\omega) \leq x \}) = F(x) \). They are easy to verify.

(b) Verify the same conditions. Use the property of the quantile function.

2. Clearly, \( (A_n, B_n) \to_p (0, 0) \). From the Slutsky’s theorem,

\[
(X_n, Y_n) + (A_n, B_n) \to_d (X, Y).
\]

I.e., \( (X_n + A_n, Y_n + B_n) \to_d (X, Y) \). Apply the continuous mapping theorem to \( g(x, y) = x/y \).
Since \( Y > 0 \), we obtain the result. The counter example is as follows: let \( X_n = X \) and \( Y_n = X \) where \( X \) has a chi-square distribution. Let \( \tilde{X} \) be another random variable independent of \( X \) but with the same distribution. Set \( A_n = B_n = 0 \). Then \( X_n \to_d X \) and \( Y_n \to_d \tilde{X} \) but

\[
\frac{X_n}{Y_n} = 1 \text{ does not converge in distribution to } \frac{X}{\tilde{X}}.
\]

3. (a) The proof that \( \mathcal{B} \cap \Omega \) is closed under complement and countable union is as follows: if \( B \cap \Omega \) is in the class, then

\[
(B \cap \Omega)^c[\text{note this is the complement in } \Omega] = B^c \cap \Omega
\]

is also in since \( B^c \in \mathcal{B} \); if \( B_1 \cap \Omega, B_2 \cap \Omega, \ldots \) are in the class, then

\[
\bigcup_n \{B_n \cap \Omega\} = \{\bigcap_n B_n\} \cap \Omega
\]

is in since \( \bigcap_n B_n \in \mathcal{B} \). Obviously, \( \lambda \times \lambda(\Omega) = 1 \).

(b) For any \( z \in R \),

\[
\{(x, y) : Z(x, y) \leq z\} = \{(x, y) : y/x \leq z\} \cap \Omega.
\]

Since \( \{(x, y) : y/x \leq z\} \) is in \( \mathcal{B} \), \( \{(x, y) : Z(x, y) \leq z\} \) is in \( \mathcal{B} \cap \Omega \). Thus, \( Z \) is measurable.

(c) \( \mu_Z \) is the Lebesgue-Stieljes measure generated by the distribution function of \( Z \), \( F_Z(z) \).

Note that for \( z \leq 0 \), \( F_Z(z) = 0 \); for \( z > 0 \),

\[
F_Z(z) = \lambda \times \lambda \{(x, y) : y \leq xz, (x, y) \in \Omega\} = \int_0^1 \int_0^{1/(xz)} I(y \leq xz) dy dx = \int_0^1 \min(1, xz) dx.
\]

The latter is equal to \( z/2 \) if \( z \leq 1 \) and is equal to \( 1 - 1/(2z) \) if \( z > 1 \).

(d) Since \( F_Z \) has no discontinuous point, the dominating measure is the Lebesgue measure (this requires verifying the condition of absolute continuity). The density is

\[
f_Z(z) = \frac{1}{2} I(0 < z \leq 1) + \frac{1}{2z^2} I(z > 1).
\]

(e) From the density, we obtain

\[
E[Z] = \int_0^1 \frac{z}{2} dz + \int_1^\infty \frac{1}{2z^2} dz = \infty.
\]

(f) Define a new random variable \( Y \) as \( Y(x, y) = y \). One can easily check that \( Y \) and \( W \) are independent and both have uniform distribution in \([0, 1] \). Clearly, \( Z = Y/W \). Thus,

\[
\]

An alternative way is to find the joint density of \((W, Z)\) then compute the conditional expectation.
4. (a) Since $\text{Cov}(X,Y - cX) = \text{Cov}(X,Y) - c\text{Cov}(X,X) = \rho - c$, we conclude that if $c = \rho$, $X$ and $Y - \rho X$ are independent.

(b) Let $Z = Y - \rho X$. The calculation is the following:

$$E[X^2Y^2] = E[X^2(\rho X + Z)^2] = \rho^2 E[X^4] + 2\rho E[X^3Z] + E[X^2Z^2].$$

Since $X$ and $Z$ are independent and with normal distributions with mean zeros, moreover, $Z$’s variance is equal to $(1 - \rho^2)$, we obtain

$$E[X^2Y^2] = 3\rho^2 + (1 - \rho^2) = 1 + 2\rho^2.$$

(c) The MGF of $(X,Y)$ is given by

$$m(t,s) = \exp \left\{ (t,s) \left( \begin{array}{c} 1 \\ \rho \\ 1 \end{array} \right) \left( t,s \right)'/2 \right\} = \exp \left\{ t^2/2 + s^2/2 + pts \right\}.$$

By differentiation, the coefficient of $t^2s^2/(2!)^2$ in the Taylor expansion of $m(t,s)$ is given by $(1 + 2\rho^2)$.

5. (a) The joint distribution is as follows: for $x, y \in (0, 1)$ and $x \leq y$,

$$P(X_{(1)} > x, X_{(n)} \leq y) = P(x < X_1 \leq y, ..., x < X_n \leq y) = (y - x)^n.$$

Thus the joint density is equal to $n(n - 1)(y - x)^{n-2}I(0 < x < 1, x \leq y < 1)$.

(b) $E[X_{(1)}|X_{(n)} = y] = \int_x x(y - x)^{n-2}I(0 < x < 1, y \leq x \leq y < 1)dx = \frac{y}{n}.$

(c) Using the transformation from $(X_{(1)}, X_{(n)})$ to $(X_{(1)} = x, X_{(n)} - X_{(1)} = z)$, we obtain that the latter has a joint density

$$n(n - 1)z^{n-2}I(0 < x < 1, 0 \leq z < 1 - x).$$

After integrating out $x$, the density of $(X_{(n)} - X_{(1)})$ is equal to $n(n - 1)z^{n-2}(1 - z)I(0 \leq z < 1)$, i.e., Beta-distribution $Beta(n - 1, 2)$.

6. (a) Since $V = \{|U| + X + Y\}/2$, we obtain

$$X = V + (-|U| + U)/2, \quad Y = V + (-|U| - U)/2.$$

(b) The support for $(X,Y)$ is $\{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Thus, the support of $(U,V)$ is $0 \leq v + (-|u| + u)/2 \leq 1, \quad 0 \leq v + (-|u| - u)/2 \leq 1.$

Clearly, $|u| \leq 1$. Then if $u < 0$, the above inequalities give $-u \leq v \leq 1$; if $u \geq 0$, we obtain $u \leq v \leq 1$. Hence, the support of $(U,V)$ should be $\{(u,v) : |u| \leq v \leq 1\}$.

(c) Note $V = \{|X - Y| + (X + Y)\}/2$. Thus, the Jacobian of $(U,V)$ with respect to $(X,Y)$ is given by

$$\left| \begin{array}{cc} 1 & -1 \\ \frac{\text{sgn}(x-y) + 1}{2} & \frac{-\text{sgn}(x-y) + 1}{2} \end{array} \right| = 1.$$

Then the Jacobian $\frac{\partial(X,Y)}{\partial(U,V)}$ is equal to $1$. From the previous support calculation, we conclude that the joint density of $(U,V)$ is equal to

$$f(u,v) = I \{|u| \leq v \leq 1\}.$$
(d) Obviously, $U$ and $V$ are not independent but since $E[U|V] = 0$,


(e) The marginal density of $V$ is equal to $2vI(0 \leq v \leq 1)$. Thus,

$$E[U^2|V = 0.5] = \int_{-\infty}^{\infty} u^2 I\{|u| \leq 1, |v| \leq 1\} du \bigg|_{v=0.5} = \frac{1}{12}.$$ 

7. (a) From $\text{Cov}(\tilde{\epsilon}_i, X_i) = \rho \sigma - c$, we conclude that if $c = \rho \sigma$, $\tilde{\epsilon}_i$ is independent of $X_i$.

(b) The expectation $\hat{\beta}_n$ is $\beta_0 + \rho \sigma$. This is from the following calculation

$$E[\hat{\beta}_n] = E\left[ \frac{\sum_{i=1}^{n} X_i((\beta_0 + \rho \sigma)X_i + \tilde{\epsilon}_i)}{\sum_{i=1}^{n} X_i^2} \right] = E\left[ \beta_0 + \rho \sigma + \frac{\sum_{i=1}^{n} X_i \tilde{\epsilon}_i}{\sum_{i=1}^{n} X_i^2} \right] = \beta_0 + \rho \sigma.$$ 

Clearly, $\hat{\beta}_n$ is an unbiased estimate of $\beta_0$ if and only if $\rho \sigma = 0$, i.e., $X_i$ and $\epsilon_i$ are independent.

(c) From the previous part, we know

$$\hat{\beta}_n = \beta_0 + \rho \sigma + \frac{\sum_{i=1}^{n} X_i \tilde{\epsilon}_i}{\sum_{i=1}^{n} X_i^2} \equiv \beta_0 + \rho \sigma + \frac{Q_n}{P_n},$$

where

$$Q_n = \frac{\sum_{i=1}^{n} X_i \tilde{\epsilon}_i}{\sqrt{\sum_{i=1}^{n} X_i^2}}, \quad P_n = \sum_{i=1}^{n} X_i^2.$$ 

Note that conditional on $X_1, ..., X_n$,

$$Q_n = \frac{\sum_{i=1}^{n} X_i \tilde{\epsilon}_i}{\sqrt{\sum_{i=1}^{n} X_i^2}} \sim N(0, (1 - \rho^2)\sigma^2).$$

Thus, $Q_n$ is independent of $(X_1, ..., X_n)$ so is independent of $P_n$. Moreover, $P_n$ has a Chi-square distribution with $n$ degrees of freedom. Then

$$\frac{Q_n}{P_n} \sim \sqrt{\frac{(1 - \rho^2)\sigma^2}{n}} t(n)$$

where $t(n)$ denotes the $t$-distribution with $n$ degrees of freedom. Hence,

$$\hat{\beta}_n \sim \beta_0 + \rho \sigma + \frac{1}{\sqrt{n}} t(n) \sqrt{(1 - \rho^2)\sigma^2},$$

which is a shifted and scaled $t$-distribution with $n$ degrees of freedom.

8. (a) The $(X, Y)$-induced measure, $\mu_{(X,Y)}$, is the Lebesgue-Stieltjes measure generated by the joint distribution of $(X, Y)$. The joint distribution function of $(X, Y)$ is the given by $\Phi(x)\Phi(y)$, where $\Phi$ is the cumulative normal distribution function. Specifically, for any Borel set $B$ in $R^2$,

$$\mu_{(X,Y)}(B) = \int_{(x,y)\in B} \phi(x)\phi(y)d\lambda(x)d\lambda(y).$$
(b) Since \( \{ Z \leq z \} = \{ X \leq z \} \cap \{ Y \leq z \} \), the conclusion is clear.

(c) Denote \( E[X|Z] = g(Z) \). From the equation

\[
E[I(Z \leq z)g(Z)] = E[I(Z \leq z)X] = E[XI(X \leq z)I(Y \leq z)],
\]

we obtain

\[
\int_{-\infty}^{z} g(z)f_Z(z)dz = \int_{-\infty}^{x} x\phi(x)d\Phi(z),
\]

where \( f_Z(z) \) is the density of \( Z \) given by \( d/dz(\Phi(z)^2) \). Differentiating both sides with respect to \( z \), we have

\[
g(z) = \frac{z\phi(z)\Phi(z) + \int_{-\infty}^{z} x\phi(x)dx\phi(z)}{2\phi(z)\Phi(z)} = \frac{z}{2} + \frac{1}{2} \int_{z}^{\infty} x\phi(x)dx \Phi(z).
\]

9. It suffices to show that for any subsequence of \( X_n \), there exists a further subsequence such that \( E[|X_n - X|] \rightarrow 0 \). First, since \( X_n \rightarrow_p X \), there exists a further subsequence such that \( X_n \rightarrow_{a.s.} X \), still denoted as \( X_n \). Apply the Fatou’s lemma to \( |X_n| + |X| - |X_n - X| \) then we obtain

\[
E\left[ \lim inf_n \{ |X_n| + |X| - |X_n - X| \} \right] \leq \lim inf_n \{ E[|X_n|] + E[|X|] - E[|X_n - X|] \}.
\]

The left-hand side is equal to \( 2E[|X|] \) while the right-hand side is equal to \( 2E[|X|] - \lim sup_n E[|X_n - X|] \). Thus,

\[
\lim sup_n E[|X_n - X|] \leq 0.
\]

The conclusion holds.