Practice Problems for Midterm 2007

1. Let F be an arbitrary distribution function in $(-\infty, \infty)$.

   (a) Define a probability measure space $(R, \mathcal{B}, \lambda_F)$ where λ_F is the Lebesgue-Stieljes measure generated by F. Show that a function defined on this space $X : R \to R$ with $X(\omega) = \omega$ is a random variable with distribution function F.

   (b) Let $([0,1], \mathcal{B} \cap [0,1], \lambda)$ be another probability measure where λ is the Lebesgue measure. Show that a function defined on this space $X : [0,1] \to R$ with $X(\omega) = F^{-1}(\omega)$ is a random variable with distribution function F.

2. Let X_n, Y_n, A_n, B_n be random variables satisfying

$$(X_n, Y_n) \to_d (X, Y), \quad A_n \to_p 0, \quad B_n \to_p 0,$$

   where $Y > 0$. Show

$$\frac{X_n + A_n}{Y_n + B_n} \to_d \frac{X}{Y}.$$

   Show by example that if $X_n \to_d X$ and $Y_n \to_d Y$ with $Y > 0$, the above result may not be true.

3. Define $\Omega = \{(x, y) : 0 < x < 1, 0 < y < 1\}$. Let $\mathcal{B}$ denote the Borel σ- field in R^2 and $\lambda \times \lambda$ be the Lebesgue measure in R^2.

   (a) Show that $\mathcal{B} \cap \Omega$ is a σ-field and $\lambda \times \lambda$ is a probability measure in Ω.

   (b) Define a function Z from Ω to $(-\infty, \infty)$ as $Z(x, y) = y/x$. Show that Z is a measurable function, i.e., a random variable.

   (c) What is the induced measure by Z in the real space, denoted by μ_Z? Give the detail of μ_Z.

   (d) Specify a measure in the real space such that μ_Z is absolutely continuous with respect to it and find the density of Z. No justification is necessary.

   (e) Use the above density to calculate $E[Z]$.

   (f) If we define another random variable W from Ω to $(-\infty, \infty)$ as $W(x, y) = x$, then what is the conditional expectation of Z given W, $E[Z|W]$?

4. Assume that X and Y are two normal random variables with mean zeros and variance ones. The correlation coefficient between X and Y is ρ.

   (a) Find a constant c so that X and $Y - cX$ are independent.

   (b) Use the previous result to calculate $E[X^2Y^2]$.

   (c) What is the moment generating function of (X, Y)? Show how to use this function to calculate $E[X^2Y^2]$.

5. Let $X_1, ..., X_n$ be i.i.d from Uniform(0,1), where $n \geq 2$. Denote $X_{(1)}$ and $X_{(n)}$ are the minimum and the maximum of $X_1, ..., X_n$ respectively.

   (a) Find the joint distribution of $(X_{(1)}, X_{(n)})$.

   (b) What is the conditional expectation $E[X_{(1)}|X_{(n)}]$?

   (c) What is the distribution of $(X_{(n)} - X_{(1)})$?

6. Let X and Y be independent Uniform(0,1) random variables. Define $U = X - Y$ and $V = \max(X, Y)$.

(a) Use the fact that $\max(x, y) = \{|x - y| + x + y\}/2$. Express X and Y in terms of U and V.

(b) What is the support of (U, V)? Simplify your answer as much as possible.

(c) Find the joint density of (U, V). (*Hint:* $d|x|/dx = sign(x)$)

(d) What is $Cov(U, V)$? Are U and V independent?

(e) What is the conditional expectation $E[U^2|V = 0.5]$?

7. Suppose that n pairs of observations (Y_1, X_1),...., (Y_n, X_n) are i.i.d generated from the following model:

$$Y_i = \beta_0 X_i + \epsilon_i, \quad i = 1, ..., n$$

where β_0 is a constant and (X_i, ϵ_i) follows a bivariate normal distribution with mean zeros and covariance

$$\begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}.$$

Here, ρ is the correlation coefficient for (X_i, ϵ_i) and σ^2 is the variance of ϵ_i. It is well known that the coefficient from Y_i regressing on X_i should be

$$\hat{\beta}_n = \sum_{i=1}^{n} X_i Y_i / \sum_{i=1}^{n} X_i^2.$$

(a) Define a new random variable $\tilde{\epsilon}_i = \epsilon_i - cX_i$ then the model can be rewritten as

$$Y_i = (\beta_0 + c)X_i + \tilde{\epsilon}_i.$$

Find a constant c so that $\tilde{\epsilon}_i$ is independent of X_i.

(b) Calculate the expectation of $\hat{\beta}_n$. When is $\hat{\beta}_n$ an unbiased estimate of β_0?

(c) What is the distribution of $\hat{\beta}_n$?

8. Suppose that (X, Y) is a bivariate random variables defined on a probability measure space $(\Omega, \mathcal{A}, P)$. Moreover, X and Y are independent and each follows a standard normal distribution.

(a) What is the (X, Y)-induced measure? This should be a measure for the Borel sets in R^2.

(b) Define $Z = \max(X, Y)$. Show that Z is also a measurable function (equivalently, a random variable).

(c) Calculate the conditional expectation $E[X|Z]$. You may keep the integration in the final expression.

9. Suppose that $X_1, X_2, ...$ are a sequence of random variables satisfying $E[|X_n|] = 1$. Moreover, X_n converges in probability to a random variable X with $E[|X|] = 1$. Prove the following convergence:

$$E[|X_n - X|] \to 0.$$

(*Hint: Apply the Fatou's lemma to* $|X_n| + |X| - |X_n - X|$)