BIOS 760 MIDTERM II, 2017: Solution

1. (5 points) The left-hand-side can be re-expressed as $n^{-1}\sum_{i=1}^{m} X_i^2 - (\bar{X}_n)^2$. Since the required moments exist, $n^{-1}\sum_{i=1}^{n} X_i^2 \to_{a.s.} \sigma^2 + \mu^2$ and $\bar{X}_n \to_{a.s} \mu$, and the conclusion follows from the continuous mapping theorem.

2. (a) (5 points) The form of the kernel follows from the definitions. For the expectation,

$$\begin{aligned} Eh^2((X_1,Y_1),(X_2,Y_2)) &= \frac{1}{4}E\left[(X_1-X_2)^2(Y_1-Y_2)^2\right] \\ &\leq \left[E(X_1-X_2)^4 E(Y_1-Y_2)^4\right]^{1/2} \\ &= \|X_1-X_2\|_4^2 \cdot \|Y_1-Y_2\|_4^2 \\ &\leq 16\|X_1-\mu_x\|_4^2 \cdot \|Y_1-\mu_y\|_4^2 \\ &< \infty, \end{aligned}$$

where $\|U\|_r \equiv (E|U|^r)^{1/r}$, for any $r \geq 1$; the first inequality follows for Holder's inequality; and the second inequality follows from Minkowski's inequality.

(b) (5 points) We first verify that $EU_n = v$. This follows since

$$E[(X_1-X_2)(Y_1-Y_2)] = E[(X_1-\mu_x)(Y_1-\mu_y)] + E[(X_2-\mu_x)(Y_2-\mu_y)] = 2v,$$

where the first equality follows from subtracting and adding μ_x from X_1 then X_2, doing the same with Y_1 and Y_2 using μ_y, and then utilizing the independence of the pairs (X_1,Y_1) and (X_2,Y_2); and the second equality follows from the definition of v. We then use 2(a) to verify that the conditions of the U-statistic CLT are satisfied with limiting variance of the given form, except that r^2 is in the place of 4. However, since $r=2$, we have that $r^2=4$, and desired conclusion follows.

(c) (5 extra credit points) We first note that under independence, $v=0$, and thus $U_n - v = U_n$. We then compute

$$\begin{aligned} &\text{cov}\left[h((X_1,Y_2),(X_2,Y_2)), h((X_1,Y_2),(\tilde{X}_2,\tilde{Y}_2))\right] \\ &= \frac{1}{4}E\left[(X_1-X_2)(Y_1-Y_2)(X_1-\tilde{X}_2)(Y_1-\tilde{Y}_2)\right] \\ &= \frac{1}{4}E\left[(X_1-X_2)(X_1-\tilde{X}_2)\right]\cdot E\left[(Y_1-Y_2)(Y_1-\tilde{Y}_2)\right] \\ &= \frac{1}{4}E[(X_1-\mu_x)^2]\cdot E[(Y_1-\mu_y)^2] \\ &= \frac{\sigma_x^2\sigma_y^2}{4}, \end{aligned}$$

where the second inequality follows from the independence assumptions, and the third inequality follows from being able to add and subtract means as in 2(b) combined with reapplication of the independence assumptions. Now the 4 and 1/4 cancel, and we can use the the U-statistic CLT result from 2(b) to verify that

$$\frac{\sqrt{n}U_n}{\sigma_x \sigma_y} \to_d N(0,1).$$

Now the final conclusion follows by using the results from problem 1 to verify that both $S_x \to_p \sigma_x$ and $S_y \to_p \sigma_y$, followed with an application of Slutsky's Theorem.

3. (5 points) First we note that since $\sigma(X_n) \subset \mathcal{F}_n$, X_n is adapted to $\mathcal{F}_n$ for all $n \geq 1$. Next we compute

$$\begin{aligned} E[X_n|\mathcal{F}_{n-1}] &= \frac{1}{3}E\left[(X_{n-1}+Z_n)^3 - X_{n-1}^3|X_{n-1}\right] \\ &= \frac{1}{3}\left[3X_{n-1}^2 E[Z_n] + 3X_{n-1}E[Z_n^2] + E[Z_n]^3\right] \\ &= \frac{3}{3}X_{n-1} \\ &= X_{n-1}, \end{aligned}$$

for all $n \geq 2$, where the first equality follows from the fact that X_{n-1} is the only part of X_n contained within $\mathcal{F}_n$; and the second equality follows from the properties of the first three moments of a standard normal random variable. Thus $(X_n, \mathcal{F}_n)$ satisfies the properties of a martingale. By applying this recursively, we obtain that $EX_n = EX_1$ for all $n \geq 1$. Reapplication of the properties of the first three moments of a standard normal random variable now yields the final conclusion.