BIOS 760 MIDTERM II, 2017

1. (5 points) Let $X_1, \ldots, X_n$ be i.i.d. real random variables with finite mean μ and finite variance σ^2. Shoe that
$$n^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \to \sigma^2$$
almost surely, as $n \to \infty$.

2. Assume the pairs of real random variables $(X_1, Y_1), \ldots, (X_n, Y_n)$ are i.i.d., with $EX_1 = \mu_x$, $EY_1 = \mu_y$, $\text{var}(X_1) = \sigma_x^2$, $\text{var}(Y_1) = \sigma_y^2$, and $\text{cov}(X_1, Y_1) = v$, with $E(X_1 - \mu_x)^4 < \infty$ and $E(Y_1 - \mu_y)^4 < \infty$. Define
$$U_n = \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \frac{(X_i - X_j)(Y_i - Y_j)}{2}.$$
Do the following:
   (a) (5 points) Explain why U_n is a second order U-statistic with kernel
   $$h((X_1, Y_1), (X_2, Y_2)) = \frac{(X_1 - X_2)(Y_1 - Y_2)}{2},$$
   and verify that $Eh^2((X_1, Y_1), (X_2, Y_2)) < \infty$.
   (b) (5 points) Verify that $\sqrt{n}(U_n - v) \to_d N(0, \tau^2)$, where
   $$\tau^2 = 4\text{cov}\left[h((X_1, Y_1), (X_2, Y_2)), h((X_1, Y_1), (\tilde{X}_2, \tilde{Y}_2))\right],$$
   where the pair $(\tilde{X}_2, \tilde{Y}_2)$ has the same joint distribution as (X_1, Y_1) but is independent of both (X_1, Y_1) and (X_2, Y_2).
   (c) (5 extra credit points) Assume now that X_i and Y_i are independent for all $i \ge 1$, and define $S_x^2 = n^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ and $S_y^2 = n^{-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2$. Show that
   $$\frac{\sqrt{n}U_n}{S_x S_y} \to_d N(0, 1),$$
   as $n \to \infty$.

3. (5 points) Let $Z_1, Z_2, \ldots$ be i.i.d. $N(0, 1)$. Define $X_1 = \left[(1 + Z_1)^3 - 1\right]/3$ and, for all $n \ge 1$, also define $X_{n+1} = \left[(X_n + Z_{n+1})^3 - X_n^3\right]/3$. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that $(X_n, \mathcal{F}_n)$ is a martingale and that $EX_n = 1$ for all $n \ge 1$.