Solution to BIOS 760 Midterm Exam II (Fall, 2016)

1. (a) (2 points) The Slutsky's theorem.

   (b) (3 points) Let $X_n = Z \sim N(0,1)$ and $Y_n = Y$ independent of Z. Then $X_n \to_d -Y$ but $X_n + Y_n \sim N(0,2)$ and $X + Y = 0$.

   (c) (5 points) Consider the characteristic functions and we obtain $\phi_{X_n+Y}(t) = E[e^{it(X_n+Y)}] = E[e^{itX_n}]E[e^{itY}] \to E[e^{it\tilde{X}}]E[e^{itY}] = E[e^{it(\tilde{X}+Y)}]$.

2. (a) (3 points) $var(\bar{Y}_n) = n^{-2}\sum_{k=1}^n \sigma_k^2 \leqslant M^2/n \to 0$. $Y_n \to_p \mu$ follows from Chebyshev's inequality.

   (b) (2 points)
$$\frac{\bar{Y}_n - \mu}{\sqrt{n^{-2}\sum_{k=1}^n \sigma_k^2}} = \sum_{k=1}^n \frac{\sigma_k}{\sqrt{\sum_{k=1}^n \sigma_k^2}} Z_k.$$
   Since
$$max_k \frac{\sigma_k}{\sqrt{\sum_{k=1}^n \sigma_k^2}} \leqslant \frac{M}{\sqrt{nm^2}} \to 0,$$
   the weighted CLT gives
$$\frac{\bar{Y}_n - \mu}{\sqrt{n^{-2}\sum_{k=1}^n \sigma_k^2}} \to_d N(0,1).$$

   (c) (2 points)
$$var(Y_{wn}) = \sum_{k=1}^n w_k^2 \sigma_k^2.$$
   Minimizing $var(Y_{wn})$ subject to constraint $\sum_k w_k = 1$ gives the optimal weights to be
$$w_k = \frac{1/\sigma_k^2}{\sum_{k=1}^n 1/\sigma_k^2}.$$

   (d) (3 points)
$$\frac{Y_{wn} - \mu}{\sqrt{var(Y_{wn})}} = \sum_{k=1}^n \frac{\sigma_k^{-1}}{\sqrt{\sum_{k=1}^n \sigma_k^{-2}}} Z_k.$$
   It converges in distribution to $N(0,1)$ follows the weighted CLT since
$$\max_k \frac{\sigma_k^{-1}}{\sqrt{\sum_{k=1}^n \sigma_k^{-2}}} \leqslant \frac{m^{-1}}{\sqrt{nM^{-2}}} \to 0.$$