BIOS 760 Midterm Exam II (Fall, 2016)

1. Suppose $X_n \to_d X$ and $Y_n \to_p Y$, where $X_n, Y_n, n = 1, 2, ...,$ X and Y are random variables.

   (a) (2 points) If Y is a constant, state the name of the theorem to conclude $X_n + Y_n \to_d X + Y$.

   (b) (3 points) Now suppose that Y is independent of X_n for any n. Show by example that $X_n + Y_n \nrightarrow_d X + Y$.

   (c) (5 points) Under the same condition in (b), show that for any random variable $\widetilde{X}$ which has the same distribution as X and is independent of Y, it holds $X_n + Y_n \to_d \widetilde{X} + Y$.

2. Let $Z_1, Z_2, ...$ be i.i.d random variables with mean zeros and variance 1. Assume $Y_n = \mu + \sigma_n Z_n$, where μ and σ_n are constants and σ_n satisfies $0 < m \leqslant \sigma_n \leqslant M < \infty$ for two positive constants m and M.

   (a) (3 points) Consider sample mean $\bar{Y}_n = n^{-1} \sum_{k=1}^n Y_k$. Show $var(\bar{Y}_n) \to 0$. Furthermore, $\bar{Y}_n \to_p \mu$.

   (b) (2 points) Derive the asymptotic distribution of $\bar{Y}_n$ after proper normalization. Justify it.

   (c) (2 points) Consider weighted mean $Y_{wn} = \sum_{k=1}^n w_k Y_k$ where $w_1, ..., w_n$ are non-negative constants satisfying $\sum_{k=1}^n w_k = 1$. Find that the optimal weights $(w_1, ..., w_n)'$ such that $var(Y_{wn})$ is the smallest.

   (d) (3 points) For the weights derived in (c), derive the asymptotic distribution of Y_{wn} after proper normalization. Justify it.