BIOS 760 MIDTERM II, 2014: Solution

1. (a) For any $r > 0$,

$$E[X^r] = 2\theta^{-2}\int_0^\theta x^{1+r}dx = \frac{2\theta^{2+r}}{\theta^2(r+2)} = \frac{2\theta^r}{r+2},$$

and thus

$$\sigma^2(\theta) = \frac{9}{4}\left[\frac{2\theta^2}{4} - \left(\frac{2}{3}\theta\right)^2\right] = \frac{\theta^2}{8},$$

and the remaining conclusions follow for the CLT.

(b) For any $0 < u < n$,

$$\begin{aligned} P[n(\theta - X_{(n)}) > u] &= P[X_{(n)} < \theta - u/n] \\ &= (P[X_1 < \theta - u/n])^n \\ &= \theta^{-2n}\left(\theta - \frac{u}{n}\right)^{2n} \\ &= \left(1 - \frac{u}{\theta n}\right)^{2n} \\ &\to \exp(-2u/\theta), \end{aligned}$$

as $n \to \infty$. For $u \leq 0$, $P[n(\theta - X_{(n)}) > u] = 1$ for all $n \geq 1$.

(c) By (b) above, $\sqrt{n}(\hat{\theta}_2 - \theta) = o_P(1)$. Thus

$$\sqrt{n}\left[(\hat{\theta}_1 + \hat{\theta}_2)/2 - \theta\right] = (1/2)\sqrt{n}(\hat{\theta}_1 - \theta) + o_P(1),$$

and the remaining conclusions follow from (a) and Slutsky's Theorem.

(d) Since, by (c), $\sqrt{n}(\hat{\theta}_2 - \theta) \to_d N(0,0)$, we have that $\hat{\theta}_2$ is the most precise since, at the root-n rate, it has an asymptotic limiting variance of zero, which is stricly less than the corresponding asymptotic limiting variances of either $\sqrt{n}(\hat{\theta}_1 - \theta)$ or $\sqrt{n}(\hat{\theta}_3 - \theta)$.

(e) For the first one, we use the delta method via the map $\theta \mapsto g(\theta) = 1/\theta$. Since $\dot{g}(\theta) = -1/\theta^2$, we obtain from (b) that

$$n\left(\frac{1}{\hat{\theta}_2} - \frac{1}{\theta}\right) \to_d -\frac{1}{\theta^2} \times -U = \frac{U}{\theta^2},$$

and thus its limiting distibution is exponential with mean $(2\theta)^{-1}$. For the second one, we use (b) and the fact that $P(-U = 0) = 0$, combined with the extended continuous mapping theorem for the map $u \mapsto f(u) = 1/u$, to obtain that the asymptotic limiting distribution is $-1/U$, where U is as defined in (b).

2. (a) Since X_n given $\mathcal{F}_{n-1}$ is Poisson(λX_{n-1}), for $n > 1$, we have $E[X_n|\mathcal{F}_{n-1}] = \lambda X_{n-1}$ and $EX_1 = \lambda$, and thus, by induction, we obtain the desired result.

(b) First, it is clear that Y_n is adapted to $\mathcal{F}_n$, for all $n \geq 1$, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$. In addition,

$$E[Y_n|\mathcal{F}_{n-1}] = \lambda^{-n} E[X_n|\mathcal{F}_{n-1}] = \lambda^{-n} \lambda X_{n-1} = Y_{n-1}.$$

(c) Since $(Y_n, \mathcal{F}_n)$ is a martingale, it is also a submartingale. It is also easy to verify from (a) that $E|Y_n| = EY_n = 1$ for all $n \geq 1$. Thus, by the martingale convergence theorem, $Y_n \to_{a.s.} Y$, where $EY \leq \sup_n E|Y_n| = 1$.