BIOS 760 MIDTERM II, 2014

1. Let $X_1, \ldots, X_n$ be i.i.d. with density

$$\frac{2x}{\theta^2} 1\{0 \le x \le \theta\}.$$

Let $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ and $X_{(n)} = \max_{1 \le i \le n} X_i$. Do the following:

(a) (3 points) Let $\hat{\theta}_1 = (3/2)\bar{X}_n$ and show that

$$\sqrt{n}(\hat{\theta}_1 - \theta) \to_d N(0, \sigma^2(\theta)),$$

and give the form of $\sigma^2(\theta)$.

(b) (5 bonus points) Let $\hat{\theta}_2 = X_{(n)}$, and show that

$$n(\hat{\theta}_2 - \theta) \to_d -U,$$

where U is exponentially distributed with mean $\theta/2$. Hint: Consider deriving from first principles the limit of $P\left(n(\theta - \hat{\theta}_2) > u\right)$.

(c) (3 points) Let $\hat{\theta}_3 = (\hat{\theta}_1 + \hat{\theta}_2)/2$, and show that

$$\sqrt{n}(\hat{\theta}_3 - \theta) \to_d N\left(0, \frac{\sigma^2(\theta)}{4}\right).$$

(d) (3 points) Which estimator, $\hat{\theta}_1$, $\hat{\theta}_2$, or $\hat{\theta}_3$, is the most asymptotically precise? Why?

(e) (3 points) Derive the limiting distribution of both $n(1/\hat{\theta}_2 - 1/\theta)$ and $\left[n(\hat{\theta}_2 - \theta)\right]^{-1}$.

2. Let X_1 be Poisson with parameter λ, for some $0 < \lambda < \infty$. For each $n \ge 1$, let X_{n+1} given X_n be Poisson with parameter λX_n. Let $Y_n = \lambda^{-n} X_n$ and $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. Do the following:

(a) (2 points) Show that $EX_n = \lambda^n$.

(b) (3 points) Show that $(Y_n, \mathcal{F}_n)$ is a martingale.

(c) (3 points) Show that $Y_n \to_{a.s.} Y$, for some Y, and give an upper bound on the expectation of Y.