BIOS 760 MIDTERM II, 2012: Solution

1. (a) When $\alpha > r$,

$$E(X_1+1)^r = \int_0^\infty \alpha(x+1)^{-\alpha-1+r}dx = \left.\frac{\alpha(x+1)^{-\alpha+r}}{-\alpha+r}\right|_0^\infty = \frac{\alpha}{\alpha-r}.$$

When $\alpha \le r$, the above integral is bounded below by $\int_0^\infty \alpha(1+x)^{-1}dx = \infty$.

(b) When $\alpha > 1$, $E(X_1) = E(X_1+1) - 1 = (\alpha-1)^{-1}$ is finite, and thus the strong law of large numbers yields the desired result.

(c) We need $g(1/(\alpha-1)) = \alpha$, and thus $g(u) = 1 + 1/u$. The variance of X_1 equals

$$\begin{aligned} E(X_1+1)^2 - 2E(X_1) - 1 - (E(X_1))^2 &= \frac{\alpha}{\alpha-2} - \frac{2}{\alpha-1} - 1 - \frac{1}{(\alpha-1)^2} \\ &= \frac{\alpha}{(\alpha-1)^2(\alpha-2)} \\ &\equiv f(\alpha) \\ &< \infty \end{aligned}$$

when $\alpha > 2$. Hence we can apply the CLT to obtain that

$$\sqrt{n}(\bar{X}_n - 1/(\alpha-1)) \to_d N(0, f(\alpha)).$$

Now we apply the delta method to $g(\bar{X}_n)$ to obtain the desired weak convergence with

$$h(\alpha) = k(\alpha)\left[g'(1/(\alpha-1))\right]^2 = \frac{\alpha(\alpha-1)^2}{\alpha-2},$$

since $g'(u) = -1/u^2$.

(d) Use change of variables, with $u = \log(1+x)$, do obtain $x = e^u - 1$, $dx = e^u du$, and the desired expectation is

$$\int_0^\infty u^r \alpha e^{-\alpha u} du = \alpha^{-r} r!.$$

(e) Use the above to obtain that $E[\log(1+X_1)] = \alpha^{-1}$ and $\mathrm{var}[\log(1+X_1)] = \alpha^{-2}$, and thus the CLT yields that $\sqrt{n}(\bar{U}_n - \alpha^{-1}) \to_d N(0, \alpha^{-2})$. Now use $k(u) = u^{-1}$. Since $k'(1/\alpha) = -\alpha^2$, we obtain the desired result since $\alpha^{-2}\alpha^4 = \alpha^2$.

(f) Note that for $\alpha > 2$,

$$\frac{h(\alpha)}{\alpha^2} = \frac{\alpha(\alpha-1)^2}{\alpha^2(\alpha-2)} = \frac{\alpha^2 - 2\alpha + 1}{\alpha^2 - 2\alpha} > 1.$$

When $\alpha \leq 2$, this ratio is undefined since the variance for X_1 does not exist. Thus the estimator based on $\bar{U}_n$ is always superior to the one based on $\bar{X}_n$ in that it exists for all $\alpha > 0$ and, when both exist ($\alpha > 2$), it has lower asymptotic variance.

2. Since, for each $n \geq 1$, Y_n is a measurable function of $X_1, \ldots, X_n$, Y_n is adapted to $\mathcal{F}_n$. Moreover, since $Y_n = \left(\prod_{i=1}^n X_i\right)^2 = \prod_{i=1}^n X_i^2$, and $E(X_1)^2 = 1$,

$$E\left[Y_n | \mathcal{F}_{n-1}\right] = \left(\prod_{i=1}^{n-1} X_i^2\right) E(X_n^2) = Y_{n-1}.$$

3. Since $\log(u)$ is continuous for $u > 0$, the continuous mapping theorem yields $\log(X_n) \to_d \log(X)$, and thus Slutsky's theorem yields that

$$Y_n \log(X_n) \to_d y \log(X).$$

The desired result following by exponentiating both sides of the above and reapplying the continuous mapping theorem.

4. By the CLT,

$$\sqrt{n}\sigma^{-1} \begin{pmatrix} \bar{X}_n - \mu \\ \bar{Y}_n - \mu \end{pmatrix} \to_d \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

where $(Z_1, Z_2)^T$ has distribution

$$N\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

The observation given in the hint yields that

$$\frac{\sqrt{n}\left(\bar{X}_n \wedge \bar{Y}_n - \mu\right)}{\sigma} = \left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}\right) \wedge \left(\frac{\sqrt{n}(\bar{Y}_n - \mu)}{\sigma}\right).$$

Since the function $g(a, b) = a \wedge b$ is continuous, we can now use the continuous mapping theorem to obtain the desired result. Note that $g(a, b)$ is not differentiable at any point where $a = b$, and so the delta method cannot be applied here.