BIOS 760 MIDTERM II, 2012

1. Let $X_1, \ldots, X_n$ be i.i.d., non-negative real random variables with density

$$f(x;\alpha) = \alpha(x+1)^{-\alpha-1} I\{x \geq 0\}, \quad 0 < \alpha < \infty,$$

and let $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $\bar{U}_n = n^{-1}\sum_{i=1}^n \log(1+X_i)$. Do the following:

   (a) (2 points) Show that for every $0 < r < \infty$,

$$E(X_1+1)^r = \begin{cases} \infty, & \text{if } \alpha \leq r, \\ \frac{\alpha}{\alpha - r}, & \text{if } \alpha > r. \end{cases}$$

   (b) (2 points) Show that if $\alpha > 1$, $\bar{X}_n \to_{a.s.} (\alpha-1)^{-1}$.

   (c) (4 points) Show that $\sqrt{n}\left(g(\bar{X}_n) - \alpha\right) \to_d N(0, h(\alpha))$, when $\alpha > 2$, for $h(\alpha) = \alpha(\alpha-1)^2/(\alpha-2)$ and some real function $g(u)$, and give the form of $g(u)$.

   (d) (2 points) Show that for all $\alpha > 0$ and every integer $r \geq 0$, $E\left[\log(1+X_1)\right]^r = \alpha^{-r} r!$.

   (e) (3 points) Show that $\sqrt{n}(k(\bar{U}_n) - \alpha) \to_d N(0, \alpha^2)$, for all $\alpha > 0$ and some real function $k(u)$, and give the form of $k(u)$.

   (f) (2 points) Show that for $\alpha > 2$, $h(\alpha)/\alpha^2 > 1$. What happens when $\alpha \leq 2$? What does this say about the relative performances of $g(\bar{X}_n)$ and $k(\bar{U}_n)$?

2. (2 points) Let $X_1, \ldots, X_n$ be i.i.d. $N(0,1)$, and define $Y_n = \left(\prod_{i=1}^n X_i\right)^2$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that $(Y_n, \mathcal{F}_n)$ is a martingale.

3. (3 points) Suppose that X_n and Y_n are positive sequences of real random variables with $X_n \to_d X$ and $Y_n \to_d y$, where X is a positive random variable and y is a positive and finite constant. Show that $X_n^{Y_n} \to_d X^y$.

4. (5 bonus points) Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be an i.i.d. sequence of pairs of random variables where $E(X_1) = E(Y_1) = \mu$, $\text{var}(X_1) = \text{var}(Y_1) = \sigma^2$, the correlation between X_1 and Y_1 is $\rho \in [-1, 1]$, and where $|\mu| < \infty$ and $\sigma^2 < \infty$. Let $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $\bar{Y}_n = n^{-1}\sum_{i=1}^n Y_i$. Show that $\sqrt{n}(\bar{X}_n \wedge \bar{Y}_n - \mu)/\sigma \to_d Z_1 \wedge Z_2$, where $a \wedge b$ denotes the minimum of a and b and where (Z_1, Z_2) is bivariate normal with $E(Z_1) = E(Z_2) = 0$, $\text{var}(Z_1) = \text{var}(Z_2) = 1$, and with correlation ρ. Hint: Observe that for any increasing function $g(u)$, $g(a \wedge b) = g(a) \wedge g(b)$.