Solution to BIOS760: 2011 FALL SEMESTER MIDTERM EXAM II

1. (a) Let X_n and Y_n be independent normal distributions. Then $X_n \to_d X_1$ and $Y_n \to_d -X_1$ but $X_n + Y_n \sim N(0,2)$ does not converge to $X_1 - X_1 = 0$.

   (b) The characteristic function of (X_n, Y_n) is

$$\phi(t_1, t_2) = E[\exp\{it_1 X_n + it_2 Y_n\}] = E[e^{it_1 X_n}]E[e^{it_2 Y_n}]$$

$$\to_d E[e^{it_1 X}]E[e^{it_2 Y}] = E[\exp\{it_1 X + it_2 Y\}].$$

   Thus, $(X_n, Y_n) \to_d (X, Y)$. Since $(x, y) \to x + y$ is continuous, $X_n + Y_n \to_d X + Y$ follows form the continuous mapping theorem.

2. (a) This follows from $E[\sum_{i=1}^n w_i Y_i / x_i] = \sum_{i=1}^n w_i E[Y_i]/x_i = \beta$.

   (b) The variance is given as

$$\sum_{i=1}^n w_i^2 x_i / x_i^2 = \sum_{i=1}^n w_i^2 / x_i.$$

   We minimize the above term subject to constraint $\sum_{i=1}^n w_i = 1$. The Lagrange-mutiplier gives

$$w_i = x_i / \sum_{j=1}^n x_j.$$

   (c) For the optimal weights, we have

$$\sum_{i=1}^n w_i Y_i / x_i = \beta + \frac{\sum_{i=1}^n \epsilon_i \sqrt{x_i}}{\sum_{i=1}^n x_i}.$$

   Since $\max x_i / \sum_{j=1}^n x_j \to 0$, by the weighted CLT,

$$\frac{\sum_{i=1}^n \epsilon \sqrt{x_i}}{\sqrt{\sum_{i=1}^n x_i}} \to_d N(0,1).$$

   Thus,

$$\{\sum_{i=1}^n x_i\}^{1/2} \left[\sum_{i=1}^n w_i Y_i / x_i - \beta\right] \to_d N(0,1).$$