BIOS 760: Solution to Midterm II 2010

1. Note that $X_i = \beta X_{i-1} + \varepsilon_i = \ldots = \beta^{i-1}\varepsilon_1 + \beta^{i-2}\varepsilon_2 + \ldots + \beta^0\varepsilon_i$. Hence

$$\begin{aligned}\overline{X}_n &= \frac{1}{n}\sum_{i=1}^{n}(\beta^{i-1}\varepsilon_1 + \beta^{i-2}\varepsilon_2 + \ldots + \beta^0\varepsilon_i)\\ &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\sum_{j=i}^{n}\beta^{j-i}\\ &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\sum_{j=0}^{n-i}\beta^{j}\\ &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\left(\frac{1-\beta^{n-i+1}}{1-\beta}\right)\end{aligned}$$

2. Note that for odd n, $(-1)^{n-i+1} = -1$ for odd i-s and $(-1)^{n-i+1} = 1$ for even i-s. Subsisting in Equation 1 for odd n we have

$$\begin{aligned}\overline{X}_n &= \frac{1}{2n}\sum_{i=1}^{n}\varepsilon_i - \frac{1}{2n}\left(\sum_{i=1,3,\ldots,n}(-1)\cdot\varepsilon_i + \sum_{i=2,4,\ldots,n-1}\varepsilon_i\right)\\ &= \frac{2}{2n}\sum_{i=1,3,\ldots,n}\varepsilon_i\end{aligned}$$

The proof for even n is similar.

3. First note that $E|\varepsilon_i| < \infty$. This follows since $E|\varepsilon_i| \le E[\varepsilon^2 + 1] = (\sigma^2 + \mu^2) + 1$. Hence for every fixed $\delta > 0$

$$\begin{aligned}P\left(n^\alpha\left|\frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i\beta^{n-i+1}\right| > \delta\right) &\le \frac{n^\alpha}{\delta}E\left[\left|\frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i\beta^{n-i+1}\right|\right] && \text{(Markov inequality)}\\ &\le \frac{n^\alpha}{n(1-\beta)\varepsilon}\sum_{i=1}^{n}\beta^{n-i}E[|\varepsilon_i|] && \text{(triangle inequality)}\\ &\le n^{\alpha-1}\frac{E|\varepsilon_i|}{\delta} \to 0 && \text{for } \alpha < 1\end{aligned}$$

4. By Equation 1, we have that

$$\overline{X}_n = \frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i - \frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i\beta^{n-i+1}\,.$$

Since ε_i are i.i.d. with mean μ, we obtain from the w.l.l.n that

$$\frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i \to_p \frac{\mu}{1-\beta}\,.$$

From the previous question we obtain that

$$\frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i\beta^{n-i+1} \to_p 0\,.$$

By Slutsky's Theorem we conclude that $\overline{X}_n \to_d \mu/(1-\beta) + 0$ which is equivalent to $\overline{X}_n \to_p \mu/(1-\beta)$.

5. By Equation 1 we can write

$$\sqrt{n}\left(\overline{X}_n - \frac{\mu}{1-\beta}\right) = \sqrt{n}\left(\frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i - \frac{\mu}{1-\beta}\right) - \sqrt{n}\left(\frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i\beta^{n-i+1}\right)$$

Note that $\varepsilon_i/(1-\beta)$ are i.i.d. and have mean $\mu/(1-\beta)$ and variance $\sigma^2/(1-\beta)^2$. Hence by the CLT we obtain that

$$\sqrt{n}\left(\frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i - \frac{\mu}{1-\beta}\right) \to_d N(0, \sigma^2/(1-\beta)^2)\,.$$

By Question 3, we have that

$$\sqrt{n}\left(\frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i\beta^{n-i+1}\right) \to_p 0$$

and the result follows from Slutsky's Theorem.

6. By Question 2, for even $n = 2k$ we have

$$\overline{X}_n = \frac{1}{2k}\sum_{i=1}^{k}\varepsilon_{2i} = \frac{1}{k}\sum_{i=1}^{k}\frac{\varepsilon_{2i}}{2} \to_{a.e.} \frac{\mu}{2}\,.$$

Note that this is sum of i.i.d. random variables with expectation $\mu/2$ and variance $\sigma^2/4$. By the CLT we obtain that

$$\sqrt{n}\left(\overline{X}_n - \frac{\mu}{2}\right) = \sqrt{k}\sqrt{2}\left(\overline{X}_n - \frac{\mu}{2}\right) \to_d N\left(0, \frac{\sigma^2}{2}\right)$$

For odd $n = 2k+1$, note that

$$\frac{1}{n} = \frac{1}{2k+1} = \frac{1}{2(k+1)} + \frac{1}{2(k+1)(2k+1)}\,.$$

Hence,

$$\overline{X}_n = \frac{1}{2k+1}\sum_{i=1}^{k+1}\varepsilon_{2i-1} = \frac{1}{2(k+1)}\sum_{i=1}^{k+1}\varepsilon_{2i-1} + \frac{1}{2(k+1)(2k+1)}\sum_{i=1}^{k+1}\varepsilon_{2i-1}$$

and since the first expression is sum of i.i.d. random variables and $\frac{1}{2(k+1)(2k+1)}\sum_{i=0}^{k}\varepsilon_{2i-1} \to_p 0$ we obtain that $\sqrt{n}(\overline{X}_n - \mu/2) \to_d N(0, \sigma^2/2)$.

7. Write $g(t) = t^2$ and note that $g'(t) = 2t$. By the delta method we have that

$$\begin{aligned}\sqrt{n}\left(g(\overline{X}_n) - g\left(\frac{\mu}{(1-\beta)}\right)\right) &\to_d g'\left(\frac{\mu}{(1-\beta)}\right) N\left(0, \frac{\sigma^2}{(1-\beta)^2}\right) \\ &= N\left(0, \frac{4\mu^2\sigma^2}{(1-\beta)^4}\right)\end{aligned}$$

8. By Question 7 we see that $\sqrt{n}(\overline{X_n})^2$ converges to 0. By Question 5,

$$\frac{(1-\beta)}{\sigma}\sqrt{n X_n} \to_d N(0,1)\,.$$

Using the continuous mapping theorem we conclude that $n(1-\beta)^2\sigma^{-2}(\overline{X_n})^2$ converges to the square of standard normal, i.e., to ${\chi_1}^2$.