BIOS 760: Midterm II 2010

Suppose that $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. random variables with mean μ and variance σ^2. Define X_n as the autoregressive model sequence $X_1 = \varepsilon_1$, and for $n \geq 2$,

$$X_n = \beta X_{n-1} + \varepsilon_n \,,$$

where $-1 \leq \beta < 1$.

1. (4 points) Denote $\overline{X}_n \equiv \frac{1}{n}\sum_{i=1}^{n} X_i$. Use change of order in the summation to show that

$$\overline{X}_n = \frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i - \frac{1}{n(1-\beta)}\sum_{i=1}^{n}\varepsilon_i\beta^{n-i+1} \,. \tag{1}$$

Hint: First, show that $X_i = \varepsilon_i + \beta\varepsilon_{i-1} + \ldots + \beta^{i-1}\varepsilon_1$.

2. (3 points) Conclude that for $\beta = -1$: $\overline{X}_n = n^{-1}(\varepsilon_1 + \varepsilon_3 + \ldots + \varepsilon_n)$ for odd n, and $\overline{X}_n = n^{-1}(\varepsilon_2 + \varepsilon_4 + \ldots + \varepsilon_n)$ for even n.

3. (5 bonus points) For $-1 < \beta < 1$, show that

$$\frac{1}{n(1-\beta)}\sum_{i=1}\varepsilon_i\beta^{n-i} = o_p(n^{-\alpha}) \,,$$

for every $0 \leq \alpha < 1$. Hint: You can use the Markov inequality.

4. (5 points) For $-1 < \beta < 1$, show that $\overline{X}_n \to_p \frac{\mu}{(1-\beta)}$. Hint: Use Equation (1), the previous question, and the fact that $o_p(n^{-\alpha}) = o_p(1)$ for every $\alpha \geq 0$.

5. (4 points) For $-1 < \beta < 1$, show that

$$\sqrt{n}\left(\overline{X}_n - \frac{\mu}{1-\beta}\right) \to_d N(0, \tau^2) \,,$$

and derive τ^2.

6. (5 bonus points) For $\beta = -1$, show that

$$\sqrt{n}\left(\overline{X}_n - \frac{\mu}{2}\right) \to_d N(0, \nu^2) \,,$$

and derive ν^2. Hint: Use Question 2. Note the difference between odd and even n. Try first to show for even n.

7. (4 points) For $-1 < \beta < 1$, derive the asymptotic distribution of

$$\sqrt{n}\left((\overline{X}_n)^2 - \frac{\mu^2}{(1-\beta)^2}\right) .$$

8. (5 points) For $-1 < \beta < 1$, when $\mu = 0$, show that $\sqrt{n}(\overline{X}_n)^2 \to_d 0$ but

$$n\frac{(1-\beta)^2(\overline{X}_n)^2}{\sigma^2} \to_d {\chi_1}^2 \,.$$

Hint: Use the continuous mapping theorem.