BIOS 760 MIDTERM I, 2017: Solution

1. Let $U = \alpha' X$ and $V = \beta' X$. Since $(U', V')'$ is a Gaussian random vector, then U anc V are independent if and only if $\text{cov}(U, V) = 0$. Since $\text{cov}(U, V) = \alpha'\Sigma\beta$, the desired conclusions follow.

2. (a) Since μ is σ-finite, there exists a countable collection of sets $A_1, A_2, \ldots \in \mathcal{A}$ such that $\cup_i A_i = \Omega$ and $\mu(A_i) < \infty$ for all $i \geq 1$. Let $B_k = \{\omega : X(\omega) \leq k\}$ for k ranging over the positive integers. Then $\cup_{i,k}(A_i \cap B_k) = \Omega$ and $\nu(A_i \cap B_k) \leq \int_{A_i} k d\mu = k\mu(A_i) < \infty$, and ν is thus σ-finite. Since $\lambda_n(A) \leq \nu(A) + \mu(A)$ for any $A \in \mathcal{A}$, λ_n is also σ-finite. The finite sub-additivity followed by the Caratheodory Extension Theorem now yields that they are both measures.

   (b) Suppose that $A \in \mathcal{A}$ satisfies $\mu(A) = 0$. Then $\int_A X \int\{X \leq m\} d\mu \leq m\mu(A) = 0$. By the monotone convergence theorem, $\nu(A) = \int_A X d\mu = 0$ also. Since $\lambda_n(A) \leq \nu(A) + \mu(A) = 0$, we obtain that $\lambda_n \prec\prec \mu$. Now suppose $A \in \mathcal{A}$ satisfies $\lambda_n(A) = 0$. Since $\lambda_n(A) \geq \nu(A)$, we also have that $\nu(A) = 0$, and the desired conclusion follows.

   (c) Existence follows from the previous two parts and the Radon-Nikodym Theorem, and the forms

   $$\frac{d\nu}{d\mu} = X, \quad \frac{d\lambda_n}{d\mu} = X + \frac{n\log(1 + Y/n)}{1 + Y}, \quad \text{and} \quad \frac{d\nu}{d\lambda_n} = \frac{X}{X + n\log(1 + Y/n)/(1 + Y)},$$

   follow directly from the uniqueness conclusion of the Radon-Nikodym Theorem.

   (d) We first need to verify that $u \mapsto u\log(1 + Y/u)$ has strictly positive derivative whenever $u, Y > 0$. Thus

   $$\frac{n\log(1 + Y/n)}{1 + Y} \uparrow \frac{Y}{1 + Y},$$

   as $n \to \infty$. Now the monotone convergence theorem gives us the desired result.

   (e) By the previous results, we know that for any $A \in \mathcal{A}$, $\rho(A) \geq \lambda_n(A)$ for all $n \geq 1$. Thus $\rho(A) = 0$ forces $\lambda_n(A) = 0$ for all $n \geq 1$. Now by using the hint, we can verify that

   $$\frac{n\log(1 + Y/n)}{1 + Y} \geq \frac{Y}{(1 + Y/n)(1 + Y)} \geq \frac{Y}{(1 + Y)^2},$$

   for all $n \geq 1$. Now suppose $\lambda_n(A) = 0$ for some $n \geq 1$. Then

   $$\delta(A) \quad = \quad \int_A \frac{Y}{(1 + Y)^2} d\mu = 0 \qquad (1)$$

also. Suppose now that we also have that

$$\int_A \frac{Y}{1+Y} d\mu \quad > \quad 0. \tag{2}$$

By the monotone convergence theorem, we then have that

$$\int_A \frac{Y}{1+Y} 1\{Y \leq m\} d\mu > 0$$

for some $0 < m < \infty$. Equation (1) now implies

$$0 = \int_A \frac{Y}{(1+Y)^2} d\mu \geq \int_A \frac{Y}{(1+Y)^2} 1\{Y \leq m\} d\mu \geq \frac{1}{1+m} \int_A \frac{Y}{1+Y} 1\{Y \leq m\} d\mu > 0,$$

which is a contradiction. Thus (2) is false, and we have through the linearity of the integral that $\rho \prec\prec \lambda_n$ for all $n \geq 1$.

3. By symmetry, we have that $E[\log(X_i)|X_1X_2X_3X_4]$ is the same for each value of $1 \leq i \leq 4$. Moreover,

$$\sum_{i=1}^{4} E[\log(X_i)|X_1X_2X_3X_4] = E\left[\sum_{i=1}^{4} \log(X_i) \middle| X_1X_2X_3X_4\right] = \log(X_1X_2X_3X_4),$$

and the desired result follows.