BIOS 760 MIDTERM I, 2017

1. (5 points) Let X be a mean zero p-variate Gaussian random variable with positive semi-definite covariance matrix Σ. Let $\alpha, \beta \in R^p$, and show that $\alpha' X$ and $\beta' X$ are independent if and only if $\alpha' \Sigma \beta = 0$.

2. Let $X, Y \geq 0$ be measurable functions on a measure space $(\Omega, \mathcal{A}, \mu)$ with μ σ-finite and $X, Y < \infty$ almost everywhere. For any $A \in \mathcal{A}$, define the set functions $\nu(A) = \int_A X d\mu$ and
$$\lambda_n(A) = \int_A \left(X + \frac{n \log(1 + Y/n)}{1 + Y} \right) d\mu,$$
for all integers $n \geq 1$. Do the following:

   (a) (5 extra credit points) Show that ν and λ_n are σ-finite measures for all $n \geq 1$.

   (b) (5 points) Show that $\nu \prec\prec \lambda_n \prec\prec \mu$ for all $n \geq 1$.

   (c) (5 points) Verify existence and derive the form of the Radon-Nikodym derivatives $\frac{d\nu}{d\mu}$, $\frac{d\lambda_n}{d\mu}$, and $\frac{d\nu}{d\lambda_n}$, for all $n \geq 1$.

   (d) (5 points) Show that, for each $A \in \mathcal{A}$,
$$\lambda_n(A) \to \rho(A) = \int_A \left(X + \frac{Y}{1 + Y} \right) d\mu,$$
   as $n \to \infty$.

   (e) (5 take home extra credit points) Show that $\lambda_n \prec\prec \rho$ and $\rho \prec\prec \lambda_n$ for all $n \geq 1$. Hint: it may be helpful to use the fact that
$$\frac{u}{1 + u} < \log(1 + u) < u,$$
   for all $u > 0$.

3. (5 points) Let X_1, X_2, X_3, X_4 be i.i.d., positive and finite random variables. Show that
$$E\left[\log(X_1) \middle| \prod_{i=1}^{4} X_i \right] = \frac{\sum_{i=1}^{4} \log(X_i)}{4}.$$