Solution to BIOS 760 Midterm Exam I (Fall, 2016)

1. (a) $(\alpha' X, \beta' X)$ follows a bivariate normal distribution with mean zeros and covariance

$$\begin{pmatrix} \alpha'\Sigma\alpha & \alpha'\Sigma\beta \\ \beta'\Sigma\alpha & \beta'\Sigma\beta \end{pmatrix}.$$

(b) Let $Z = \alpha' X - (\alpha'\Sigma\beta)/(\beta'\Sigma\beta)\beta' X$. Then Z is independent of $\beta' X$ since $cov(\beta' X, Z) = 0$. Clearly,

$$\alpha' X = a\beta' X + Z, \quad a = (\alpha'\Sigma\beta)/(\beta'\Sigma\beta).$$

(c) $\alpha' X/|\beta' X| = a + Z/|\beta' X|$. Since $Z \sim \sqrt{\alpha'\Sigma\alpha - (\alpha'\Sigma\beta)^2/(\beta'\Sigma\beta)}N(0,1)$ and $|\beta' X| \sim \sqrt{\beta'\Sigma\beta}\chi^2(1)$, we obtain

$$\alpha' X/\beta' X \sim a + \frac{\sqrt{\alpha'\Sigma\alpha - (\alpha'\Sigma\beta)^2/(\beta'\Sigma\beta)}}{\sqrt{\beta'\Sigma\beta}} t(1).$$

2. (a) This is because $(\mu \times \lambda_F)(\Omega_1 \times \Omega_2) = \mu(\Omega_1) \times \lambda_F(\Omega_2) = 1$.

(b) $\{X \le x\} = \cup_{k=1}^{\infty} \{(w_1, w_2) : w_1 = k, I(w_2 \in [k, k+1)) \le x\} = \cup_k \{w_1 : w_1 = k\} \times \{w_2 : I(w_2 \in [k, k+1)) \le x\} \in 2^{\Omega_1} \times \mathcal{B}$. Thus, X is measurable.

(c) By the Fubini theorem,

$$\begin{aligned} & \int_{\Omega_1 \times \Omega_2} X(\omega_1, \omega_2) d(\mu \times \lambda_F) \\ = & \int_{\Omega_1} \int_{\Omega_2} I(w_2 :\in [w_1, w_1 + 1)) d\lambda_F(w_2) d\mu(w_1) \\ = & \int_{\Omega_1} \int_{w_1}^{w_1+1} dF(w_2) d\mu(w_1) \\ = & \int_{\Omega_1} \left\{ e^{-w_1} - e^{-w_1 - 1} \right\} d\mu(w_1) \\ = & \sum_{k=1}^{\infty} 2^{-k} \left\{ e^{-k} - e^{-k-1} \right\} \\ = & \frac{e^{-1}/2}{1 - e^{-1/2}} (1 - e^{-1}). \end{aligned}$$

(d) This is clear from $P(Y \le y, Z \le z) = \mu(Y \le y)\lambda_F(Z \le z)$.

(e) The measure induced by Y is dominated by the counting measure on $\{1, 2, ..\}$ with density $P(Y = k) = 2^{-k}$. The measured induced by Z is dominated by λ with density $e^{-x} I(x > 0)$.

(f) $E[X] = E[E[I(Z \in [Y, Y+1))|Y]] = E[e^{-Y} - e^{-(Y+1)}] = \frac{e^{-1}/2}{1-e^{-1/2}}(1 - e^{-1})$.